Multigraphs without large bonds are wqo by contraction

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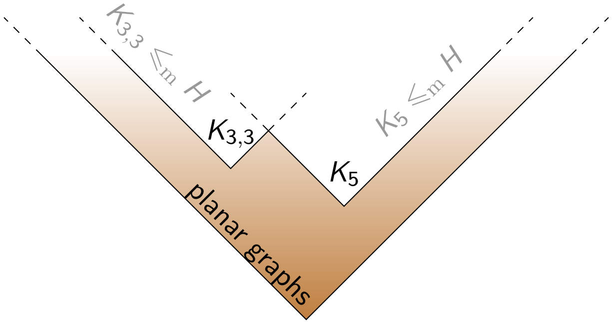
- well-quasi-order (wqo): qo with no infinite antichain and no infinite decreasing sequence;
  - $(\mathbb{N}, \leqslant)$;
  - $(\mathcal{P}(\mathbb{N}), \subseteq)$ is not a wqo $\{0\}, \{1\}, \{2\}, \dots$;
  - some containment relations on graphs.

Finite obstruction sets



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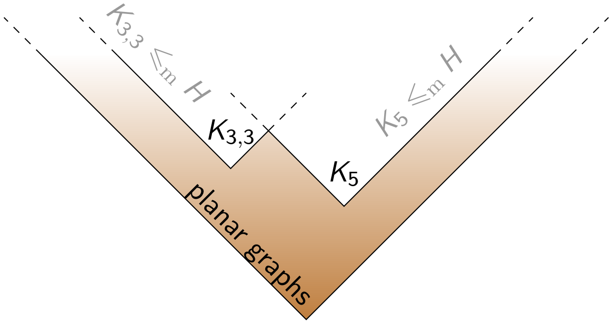

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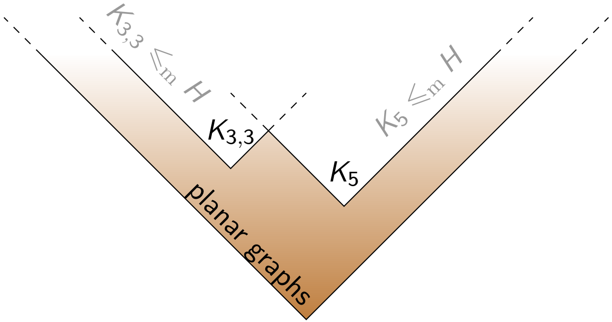

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Algorithmic consequences.

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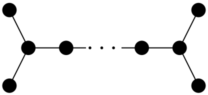

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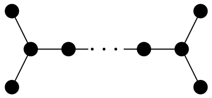

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Open problems: induced immersions and strong immersions.

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2. look at antichains (= sequences on non-comparable elements):
   - [Ding '09] $\exists$ canonical antichain for $\leqslant_{\mathrm{s}}$, but not for $\leqslant_{\mathrm{is}}$;
   - [Lozin & Mayhill '10] $\exists$ canonical antichain for $\leqslant_{\mathrm{is}}$ on unit interval graphs and on bipartite permutation graphs.

In this talk...

- multiple edges are allowed, but not loops;
- contraction of $\{u, v\}$: identifies u and v, delete loops and keeps multiple edges;
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- contraction of $\{u, v\}$: identifies u and v, delete loops and keeps multiple edges;
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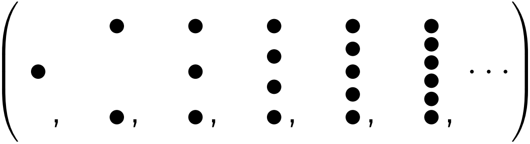
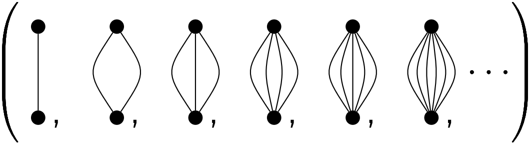
Are (finite) graphs well-quasi-ordered by $\preceq$? No!

$\mathcal{A}_{K_1} =$

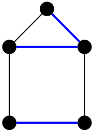


$\mathcal{A}_{\theta} =$

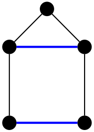


Bonds

Bond: minimal edge cut.



A bond of size 3.



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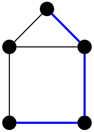

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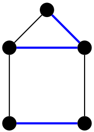

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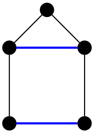

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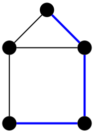

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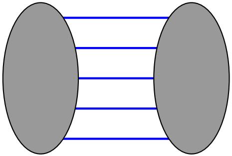

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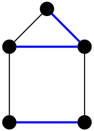

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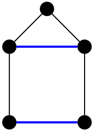

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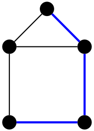
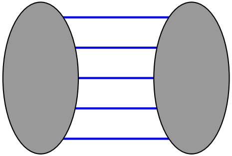


Not minimal.

- removing a bond gives 2 connected components;
- G has a bond of size k iff $\theta_k \preceq G$.



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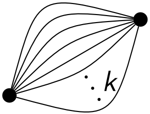

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Theorem (Kamiński, R., Trunck, 2014+)

For every $p, k \in \mathbb{N}$, the class $\mathcal{G}_{p,k}$ is well-quasi-ordered by $\preceq$.

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Corollary

$(\mathcal{H}, \preceq)$ wqo iff $\mathcal{H}$ excludes all but finitely many graphs of $\mathcal{A}_{K_1}$ and of $\mathcal{A}_\theta$.

$$\mathcal{A}_{K_1} = (K_1, 2 \cdot K_1, 3 \cdot K_1, \dots)$$
$$\mathcal{A}_\theta = (\theta_1, \theta_2, \theta_3, \dots)$$

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$$(a, b) \leqslant_{A\times B} (a', b') \text{ if } a \leqslant_A a' \text{ and } b \leqslant_B b'$$

   of two wqos is a wqo;

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   of two wqos is a wqo;
   Why?
   $X = x_1, x_2, \ldots$ infinite antichain of $(A \cup B, \leqslant_{A \cup B})$
   $\Rightarrow$ one of $X \cap A$ and $X \cap B$ is infinite.
   But both $(A, \leqslant)$ and $(B, \leqslant)$ are wqos.

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$$A = \{a, b, c, d\} \qquad abcd \leqslant_A^\star bcccd$$
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Use: build large wqos from smaller ones.

Monotonicity

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Let $(A, \leqslant_A)$ be a wqo.
Key property: if $\varphi\colon (A, \leqslant_A) \to (B, \leqslant_B)$ is monotone and surjective, then $(B, \leqslant_B)$ is a wqo.

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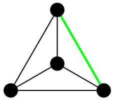
Idem for p connected components.

Structure of 2-connected graphs

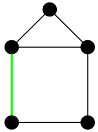
Theorem (Tutte '61)

Every 2-connected graph can be constructed from cycles and 3-connected graphs by 2-sums.

2-sum of G and H: identify $e \in \mathrm{E}(G)$ with $e' \in \mathrm{E}(H)$ and possibly delete it



G



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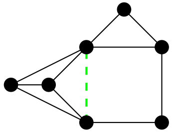

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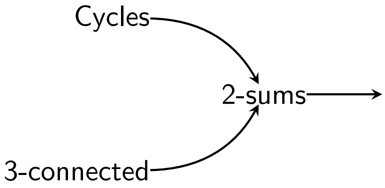

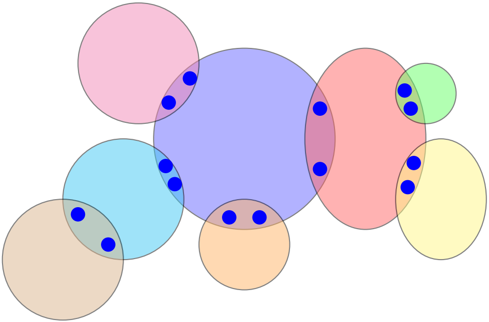

tree decomposition

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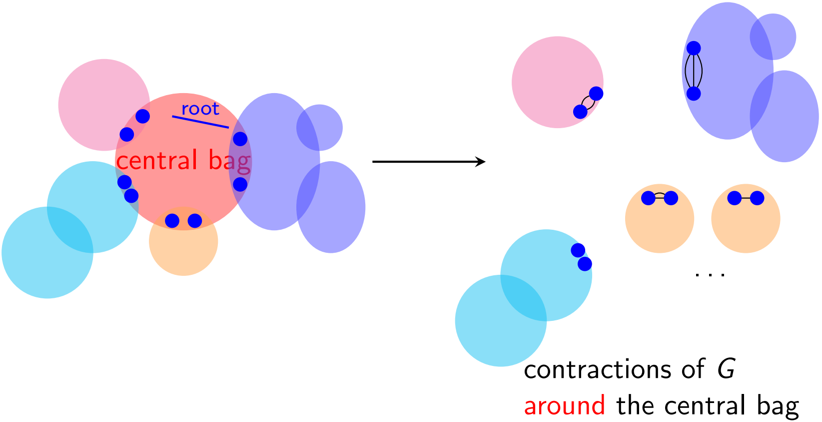

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($\forall i,\ A_i$ is minimal wrt. $\preceq$ s.t. $\exists$ an infinite antichain starting with $A_0, \ldots A_i$)

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Every graph of $\mathcal{A}$ is built using graphs of the wqo $(\mathcal{C}, \preceq)$.

Proof: 2-c graphs without big bonds are wqo by $\preceq$ (3)

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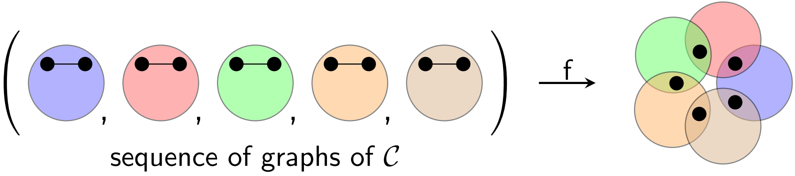
Possible cases for graphs of $\mathcal{B}$:

- graphs of $\mathcal{C}$ attached (2-sum) on the edges of cycles;
- graphs of $\mathcal{C}$ attached (2-sum) on the edges of K_4;
- ...

Proof: 2-c graphs without big bonds are wqo by $\preceq$ (4)

If graphs of $\mathcal{B}$ are constructed by attaching graphs of $\mathcal{C}$ to cycles:



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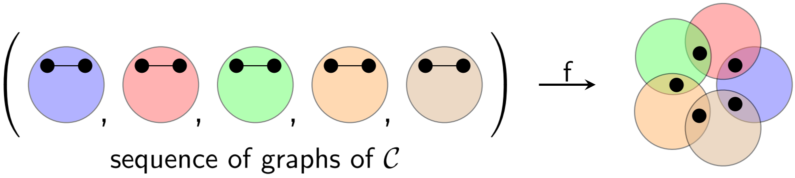

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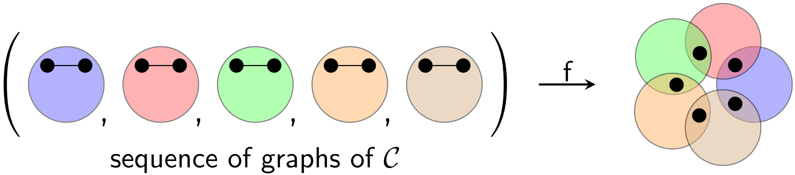

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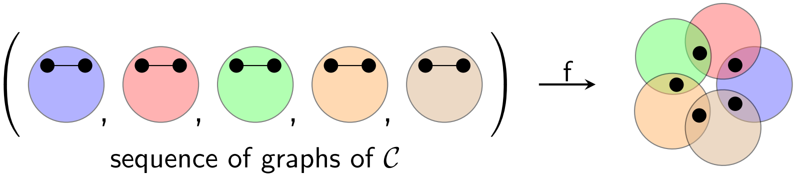

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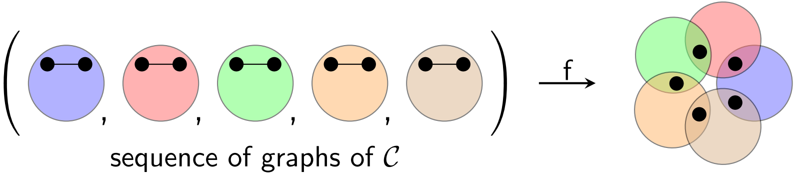

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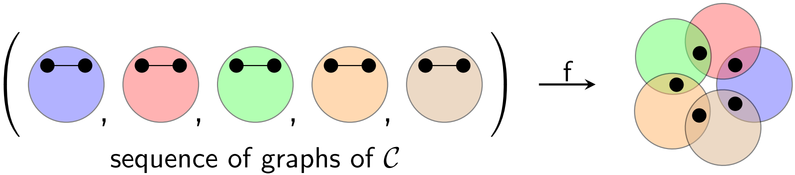

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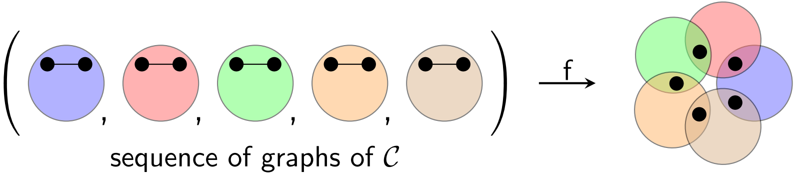

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$\rightsquigarrow$ there is no infinite antichain of edge-rooted graphs under $\preceq$

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What about infinite antichains under $\preceq$?

Looking at antichains

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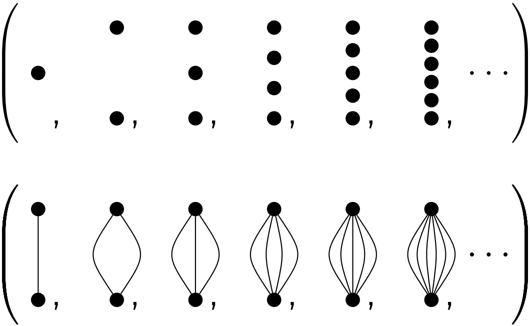

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Theorem (Kamiński, R., Trunck, 2014+)

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Corollary

There are canonical antichains for $\preceq$.

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Thank you!