An *edge* variant of Erdős–Pósa property

Jean-Florent Raymond   Ignasi Sau   Dimitrios M. Thilikos

University of Warsaw, Poland

LIRMM, Montpellier, France

National and Kapodistrian University of Athens, Greece

9th ICGT
Grenoble, June 30-July 4, 2014

Erdős–Pósa Theorem (1965)

Theorem (Erdős–Pósa)

For every $k \in \mathbb{N}$, every graph has

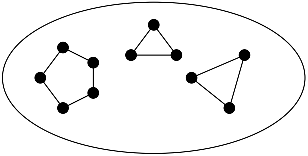
- *k vertex-disjoint cycles;*
- *or $O(k \log k)$ vertices hitting all its cycles (tight).*

Erdős–Pósa Theorem (1965)

Theorem (Erdős–Pósa)

For every $k \in \mathbb{N}$, every graph has

- *k vertex-disjoint cycles;*
- *or $O(k \log k)$ vertices hitting all its cycles (tight).*

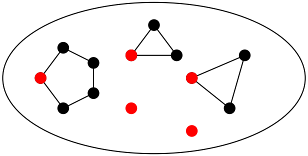


Erdős–Pósa Theorem (1965)

Theorem (Erdős–Pósa)

For every $k \in \mathbb{N}$, every graph has

- *k vertex-disjoint cycles;*
- *or $O(k \log k)$ vertices hitting all its cycles (tight).*

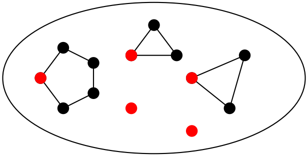


Erdős–Pósa Theorem (1965)

Theorem (Erdős–Pósa)

For every $k \in \mathbb{N}$, every graph has

- *k vertex-disjoint cycles;*
- *or $O(k \log k)$ vertices hitting all its cycles (tight).*



(relation packing / cover)

In general...

If $\mathcal{S}$ is a class of graphs, does the following hold?

$\exists f : \mathbb{N} \to \mathbb{N}$, such that $\forall k$, every G contains

- k vertex-disjoint subgraphs from $\mathcal{S}$;
- or $f(k)$ vertices hitting all the graphs of $\mathcal{S}$ that are in G.

In general...

If $\mathcal{S}$ is a class of graphs, does the following hold?

$\exists f : \mathbb{N} \to \mathbb{N}$, such that $\forall k$, every G contains

- k vertex-disjoint subgraphs from $\mathcal{S}$;
- or $f(k)$ vertices hitting all the graphs of $\mathcal{S}$ that are in G.

- If so, $\mathcal{S}$ is said to have the EP-property;
- f is the gap of $\mathcal{S}$;
- EP Theorem: cycles have the EP-property with gap $O(k \log k)$.

A few definitions

- graph $\sim$ multigraph;

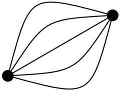
A few definitions

- graph $\sim$ multigraph;
- H-model in G : subgraph of G that can be contracted into H;

A few definitions

- graph $\sim$ multigraph;
- H-model in G : subgraph of G that can be contracted into H;
- $\mathcal{M}(H)$: class of H-models;

A few definitions

- graph $\sim$ multigraph;
- H-model in G : subgraph of G that can be contracted into H;
- $\mathcal{M}(H)$: class of H-models;
- r: multigraph with 2 vertices and r edges. 

About the Erdős–Pósa property

- König's Theorem;

About the Erdős–Pósa property

- König's Theorem;
- Erdős–Pósa Theorem;

About the Erdős–Pósa property

- König's Theorem;
- Erdős–Pósa Theorem;
- [Robertson, Seymour 86]

  $\mathcal{M}(H)$ has the EP-property $\Leftrightarrow$ H planar;

- König's Theorem;
- Erdős–Pósa Theorem;
- [Robertson, Seymour 86]
  $\mathcal{M}(H)$ has the EP-property $\Leftrightarrow$ H planar;
- several improvements of the gap function (general case);

About the Erdős–Pósa property

- König's Theorem;
- Erdős–Pósa Theorem;
- [Robertson, Seymour 86]
  $\mathcal{M}(H)$ has the EP-property $\Leftrightarrow$ H planar;
- several improvements of the gap function (general case);
- [Birmelé *et al.* 07] EP for long cycles;

About the Erdős–Pósa property

- König's Theorem;
- Erdős–Pósa Theorem;
- [Robertson, Seymour 86]
  $\mathcal{M}(H)$ has the EP-property $\Leftrightarrow$ H planar;
- several improvements of the gap function (general case);
- [Birmelé *et al.* 07] EP for long cycles;
- [Fomin *et al.* 12] f$_r(k) = O(k^2)$;

- König's Theorem;
- Erdős–Pósa Theorem;
- [Robertson, Seymour 86]
  $\mathcal{M}(H)$ has the EP-property $\Leftrightarrow$ H planar;
- several improvements of the gap function (general case);
- [Birmelé *et al.* 07] EP for long cycles;
- [Fomin *et al.* 12] f  $_r(k) = O(k^2)$;
- [Fiorini *et al.* 13] f  $_r(k) = O(k \log k)$;

About the Erdős–Pósa property

- König's Theorem;
- Erdős–Pósa Theorem;
- [Robertson, Seymour 86]
  $\mathcal{M}(H)$ has the EP-property $\Leftrightarrow$ H planar;
- several improvements of the gap function (general case);
- [Birmelé *et al.* 07] EP for long cycles;
- [Fomin *et al.* 12] f$_r(k) = O(k^2)$;
- [Fiorini *et al.* 13] f$_r(k) = O(k \log k)$;
- [Chekury, Chuzhoy 13] $\forall H$ planar, $f_H(k) = O(k \,\mathrm{polylog}\, k)$.

(non-exhaustive list)

An edge variant?

What if we replace the word "vertices" by "edges"?

An edge variant?

What if we replace the word "vertices" by "edges"?

Theorem (Erdős, Pósa, edge version)

Every graph has

- k *edge-disjoint cycles;*
- *or* $O(k \log k)$ *edges hitting all its cycles.*

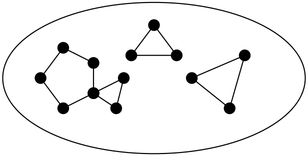
An edge variant?

What if we replace the word "vertices" by "edges"?

Theorem (Erdős, Pósa, edge version)

Every graph has

- k *edge-disjoint cycles;*
- *or* $O(k \log k)$ *edges hitting all its cycles.*



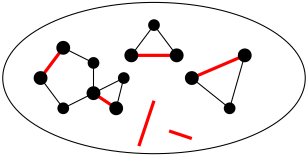
An edge variant?

What if we replace the word "vertices" by "edges"?

Theorem (Erdős, Pósa, edge version)

Every graph has

- k *edge-disjoint cycles;*
- *or* $O(k \log k)$ *edges hitting all its cycles.*

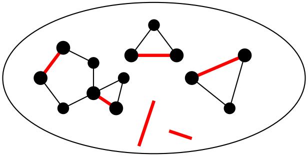


What if we replace the word "vertices" by "edges"?

Theorem (Erdős, Pósa, edge version)

Every graph has

- k *edge-disjoint cycles;*
- *or* $O(k \log k)$ *edges hitting all its cycles.*



Can we define an edge-Erdős-Pósa property?

Edge version, in general

$\mathcal{S}$ is said to have the edge-EP-property with gap f if we have

$\exists f : \mathbb{N} \to \mathbb{N}$, such that $\forall k$, every G contains

- k edge-disjoint subgraphs from $\mathcal{S}$;
- or $f(k)$ edges hitting all the graphs of $\mathcal{S}$ that are in G.

Edge version, in general

$\mathcal{S}$ is said to have the edge-EP-property with gap f if we have

$\exists f : \mathbb{N} \to \mathbb{N}$, such that $\forall k$, every G contains

- k edge-disjoint subgraphs from $\mathcal{S}$;
- or $f(k)$ edges hitting all the graphs of $\mathcal{S}$ that are in G.

Edge version of the Erdős-Pósa Theorem:
cycles have the edge-EP property with gap $O(k \log k)$.

Edge version, in general

$\mathcal{S}$ is said to have the edge-EP-property with gap f if we have

$\exists f : \mathbb{N} \to \mathbb{N}$, such that $\forall k$, every G contains

- k edge-disjoint subgraphs from $\mathcal{S}$;
- or $f(k)$ edges hitting all the graphs of $\mathcal{S}$ that are in G.

Edge version of the Erdős-Pósa Theorem:
cycles have the edge-EP property with gap $O(k \log k)$.

Nothing is known about this variant.

- We will deal with the edge-Erdős-Pósa-property.
- $S = \mathcal{M}($$)$ (Graphs that can be contracted to )

In this talk...

- We will deal with the edge-Erdős-Pósa-property.
- $S = \mathcal{M}($$)$ (Graphs that can be contracted to )

Two questions:

1. Does $\mathcal{M}($$)$ have the edge-EP-property?
2. If so, can we estimate the gap?

- We will deal with the edge-Erdős-Pósa-property.
- $S = \mathcal{M}(\theta_r)$ (Graphs that can be contracted to θ_r)

Two questions:

1. Does $\mathcal{M}(\theta_r)$ have the edge-EP-property?
2. If so, can we estimate the gap?

Remark:
$\mathcal{M}(\theta_2)$ have the edge-EP-property with gap $O(k \log k)$.
(edge version of the Erdős-Pósa Theorem)

Theorem (R., Sau, Thilikos 13)

Every graph G has

- k *edge-disjoints* θ_r*-models;*
- *or* $h_{\theta_r}(k)$ *edges hitting all its* θ_r*-models.*

$$h_{\theta_r}(k) = \begin{cases} O(k^2 r^3 \operatorname{polylog} kr) \\ O(k^4 r^2 \operatorname{polylog} kr). \end{cases}$$

Theorem (R., Sau, Thilikos 13)

Every graph G has

- *k edge-disjoints* *r-models;*
- *or h**$_r(k)$ edges hitting all its* *r-models.*

h$_r(k) = \begin{cases} O(k^2 r^3 \text{ polylog } kr) \\ O(k^4 r^2 \text{ polylog } kr). \end{cases}$

- Models of r have the edge-Erdős-Pósa property;
- The gap is polynomial **both** in k and in r.

1. re-proving the vertex version — with a gap better in r
2. from vertices to edges;
3. bounding the maximum degree — big $\Delta \Rightarrow$ big  r packing

Re-proving the vertex version

For the vertex version:

[Fomin *et al.* 12] $f_{\theta_r}(k) = O(k^2)$;

Re-proving the vertex version

For the vertex version:

[Fomin *et al.* 12] $f_{\theta_r}(k) = O(k^2)$;

[Fiorini *et al.* 13] $f_{\theta_r}(k) = O(k \log k)$;

For the vertex version:

[Fomin *et al.* 12] $f_{\theta_r}(k) = O(k^2)$;

[Fiorini *et al.* 13] $f_{\theta_r}(k) = O(k \log k)$;

We want an estimation in **both** k and r.

Lemma (R., Sau, Thilikos 13)

$\mathcal{M}(\theta_r)$ have the vertex-EP property with gap f_{θ_r} where

$$f_{\theta_r}(k) = \begin{cases} O(kr^2 \operatorname{polylog} kr) \\ O(k^3 r \operatorname{polylog} kr). \end{cases}$$

Re-proving the vertex version: bounding the treewidth

Theorem (Chekuri, Chuzhoy, 2013)

For every $k, p \in \mathbb{N}$ and G of treewidth t,

$$kp^2 \leqslant \frac{t}{\operatorname{polylog} t}$$

then $\exists G_1, G_2, \ldots, G_k$ partition of G into vertex-disjoint subgraphs and $\forall i \in \{1, \ldots, h\}$, $\mathbf{tw}(G_i) \geqslant p$.

Re-proving the vertex version: bounding the treewidth

Theorem (Chekuri, Chuzhoy, 2013)

For every $k, p \in \mathbb{N}$ and G of treewidth t,

$$kp^2 \leqslant \frac{t}{\text{polylog } t}$$

then $\exists G_1, G_2, \ldots, G_k$ partition of G into vertex-disjoint subgraphs and $\forall i \in \{1, \ldots, h\}$, $\mathbf{tw}(G_i) \geqslant p$.

- if G has big **tw**, we split it into k subgraphs of $\mathbf{tw} \geqslant 2r - 1$;

Re-proving the vertex version: bounding the treewidth

Theorem (Chekuri, Chuzhoy, 2013)

For every $k, p \in \mathbb{N}$ *and* G *of treewidth* t*,*

$$kp^2 \leqslant \frac{t}{\operatorname{polylog} t}$$

then $\exists G_1, G_2, \ldots, G_k$ *partition of* G *into vertex-disjoint subgraphs and* $\forall i \in \{1, \ldots, h\}$*,* $\mathbf{tw}(G_i) \geqslant p$*.*

- if G has big **tw**, we split it into k subgraphs of $\mathbf{tw} \geqslant 2r - 1$;

[Reed, Wood, 2012] $\mathbf{tw}(G) \geqslant 2r - 1 \Rightarrow G \leqslant_{\mathrm{m}}$  r;

Theorem (Chekuri, Chuzhoy, 2013)

For every $k, p \in \mathbb{N}$ and G of treewidth t,

$$kp^2 \leqslant \frac{t}{\text{polylog } t}$$

then $\exists G_1, G_2, \ldots, G_k$ partition of G into vertex-disjoint subgraphs and $\forall i \in \{1, \ldots, h\}$, $\mathbf{tw}(G_i) \geqslant p$.

- if G has big **tw**, we split it into k subgraphs of $\mathbf{tw} \geqslant 2r - 1$;

[Reed, Wood, 2012] $\mathbf{tw}(G) \geqslant 2r - 1 \Rightarrow G \leqslant_{\mathrm{m}}$  r;

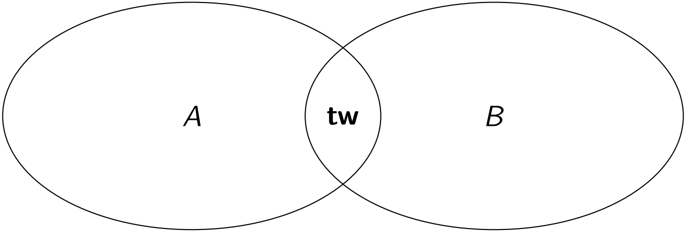
- graphs of big **tw** contain $k \cdot$  r.

We plug the bound on **tw** into the proof of [Fomin *et al.* 2012].
Idea:

Re-proving the vertex version: finding a vertex-hitting set

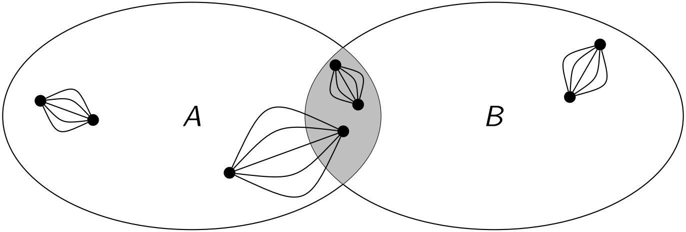
We plug the bound on **tw** into the proof of [Fomin *et al.* 2012].
Idea:

- G has a separation (A, B) of order $\mathbf{tw}(G)$
  where A has $\in [1/3, 2/3]$ of the θ_r models



We plug the bound on **tw** into the proof of [Fomin *et al.* 2012].
Idea:

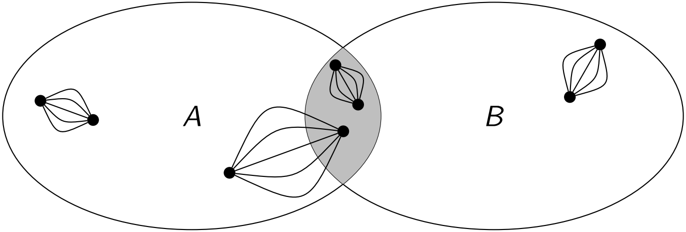
- G has a separation (A, B) of order $\mathbf{tw}(G)$
  where A has $\in [1/3, 2/3]$ of the r models
- take $A \cap B$ in the hitting set and recurse on $A \setminus B$ and $B \setminus A$
  (every model is in $A \setminus B$ or in $B \setminus A$ or uses $A \cap B$)



Re-proving the vertex version: finding a vertex-hitting set

We plug the bound on **tw** into the proof of [Fomin *et al.* 2012].
Idea:

- G has a separation (A, B) of order $\mathbf{tw}(G)$
  where A has $\in [1/3, 2/3]$ of the θ_r models
- take $A \cap B$ in the hitting set and recurse on $A \setminus B$ and $B \setminus A$
  (every model is in $A \setminus B$ or in $B \setminus A$ or uses $A \cap B$)



► gives that either $G \geqslant_{\mathrm{m}} k \cdot \theta_r$, or it has a vertex-hitting set of size $O(kr^2 \text{ polylog } kr)$.

From vertices to edges

Goal: vertex-hitting set $\to$ edge-hitting set.
$I(v)$: set of edges incident with v.

From vertices to edges

Goal: vertex-hitting set $\rightarrow$ edge-hitting set.
$I(v)$: set of edges incident with v.

A simple remark

$X \subseteq V(G)$ hits all r-models in G

From vertices to edges

Goal: vertex-hitting set $\rightarrow$ edge-hitting set.
$I(v)$: set of edges incident with v.

A simple remark

$X \subseteq V(G)$ hits all θ_r-models in G
$\Rightarrow$ every θ_r-model of G has a vertex v in X

From vertices to edges

Goal: vertex-hitting set $\to$ edge-hitting set.
$I(v)$: set of edges incident with v.

A simple remark

$X \subseteq V(G)$ hits all r-models in G
$\Rightarrow$ every r-model of G has a vertex v in X
$\Rightarrow$ every r-model of G has an edge e in $I(v)$

From vertices to edges

Goal: vertex-hitting set $\rightarrow$ edge-hitting set.
$I(v)$: set of edges incident with v.

A simple remark

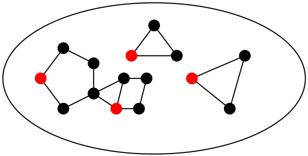
$X \subseteq V(G)$ hits all r-models in G
$\Rightarrow$ every r-model of G has a vertex v in X
$\Rightarrow$ every r-model of G has an edge e in $I(v)$
$\Rightarrow Y = \{I(v),\ v \in X\} \subseteq E(G)$ hits all r-models in G.

From vertices to edges

Goal: vertex-hitting set $\rightarrow$ edge-hitting set.
$I(v)$: set of edges incident with v.

A simple remark

$X \subseteq V(G)$ hits all θ_r-models in G
$\Rightarrow$ every θ_r-model of G has a vertex v in X
$\Rightarrow$ every θ_r-model of G has an edge e in $I(v)$
$\Rightarrow Y = \{I(v),\ v \in X\} \subseteq E(G)$ hits all θ_r-models in G.

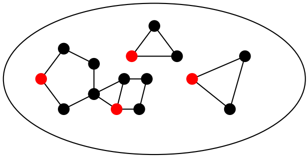


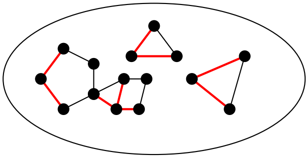
From vertices to edges

Goal: vertex-hitting set $\rightarrow$ edge-hitting set.
$I(v)$: set of edges incident with v.

A simple remark

$X \subseteq V(G)$ hits all θ_r-models in G
$\Rightarrow$ every θ_r-model of G has a vertex v in X
$\Rightarrow$ every θ_r-model of G has an edge e in $I(v)$
$\Rightarrow$ $Y = \{I(v),\ v \in X\} \subseteq E(G)$ hits all θ_r-models in G.



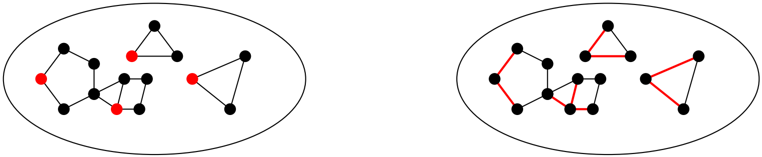


From vertices to edges

Goal: vertex-hitting set $\rightarrow$ edge-hitting set.
$I(v)$: set of edges incident with v.

A simple remark

$X \subseteq V(G)$ hits all r-models in G
$\Rightarrow$ every r-model of G has a vertex v in X
$\Rightarrow$ every r-model of G has an edge e in $I(v)$
$\Rightarrow$ $Y = \{I(v),\ v \in X\} \subseteq E(G)$ hits all r-models in G.



- vertex-hitting set of size $\ell \Rightarrow$ edge-hitting set of size $\leqslant \Delta\ell$
- Works for every class $\mathcal{S}$ of connected graphs.

By the vertex version of the theorem (for r), for every G, k,

- G contains k vertex-disjoint r-models;

By the vertex version of the theorem (for r), for every G, k,

- G contains k vertex-disjoint r-models;
  - they are also edge-disjoint;

By the vertex version of the theorem (for r), for every G, k,

- G contains k vertex-disjoint r-models;
  - they are also edge-disjoint;
- or G has a set of f$_r(k)$ vertices hitting all its r-models;

By the vertex version of the theorem (for r), for every G, k,

- G contains k vertex-disjoint r-models;
  - they are also edge-disjoint;
- or G has a set of f$_r(k)$ vertices hitting all its r-models;
  - $\Delta(G)f$$_r(k)$ edges hitting all its r-models.

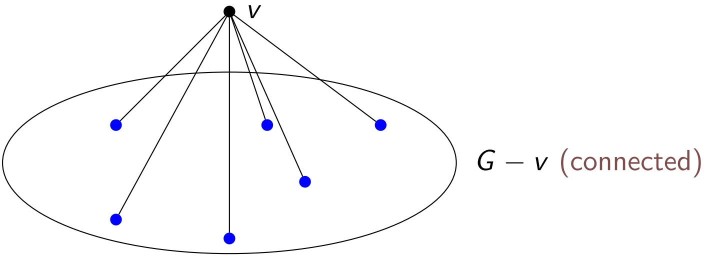
By the vertex version of the theorem (for θ_r), for every G, k,

- G contains k vertex-disjoint θ_r-models;
  - they are also edge-disjoint;
- or G has a set of $f_{\theta_r}(k)$ vertices hitting all its θ_r-models;
  - $\Delta(G) f_{\theta_r}(k)$ edges hitting all its θ_r-models.

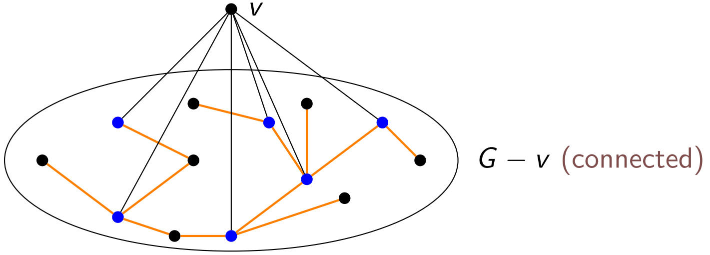
What if Δ is big?

Bounding the maximum degree

G 2-connected, v a vertex of degree $2kr$.

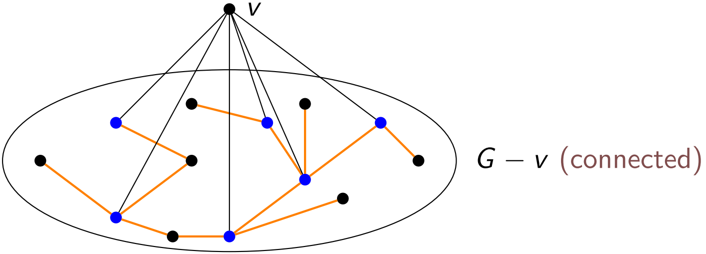


G 2-connected, v a vertex of degree $2kr$.



Bounding the maximum degree

G 2-connected, v a vertex of degree $2kr$.

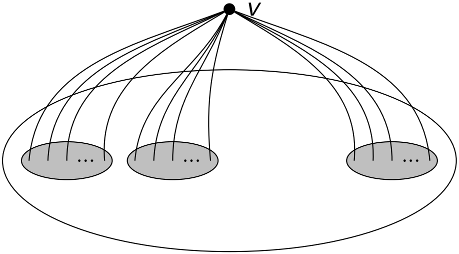


Lemma

T has k disjoint subtrees, each with r blue vertices.

Bounding the maximum degree

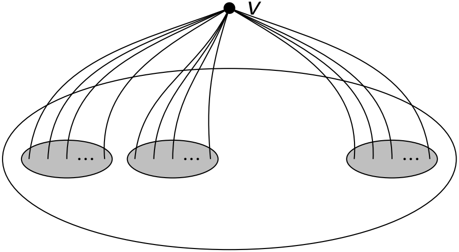
G 2-connected, v a vertex of degree $2kr$.



Lemma

T has k disjoint subtrees, each with r blue vertices.

G 2-connected, v a vertex of degree $2kr$.



$G - v$ (connected)

Lemma

T has k disjoint subtrees, each with r blue vertices.

Lemma

If $\Delta(G) \geqslant 2kr$ then G has k edge-disjoint  *r-models.*

Bounding the maximum degree (2)

If G does not have k edge-disjoint r-models,

- $\Delta(G) < 2kr$;
- G has $\Delta(G)f$$_r(k) < 2kr \cdot f$$_r(k)$ edges hitting all its r-models.

▶ this is the edge-Erdős-Pósa property.

The edge-Erdős-Pósa property

Known:

- cycles: gap $O(k \log k)$;

Known:

- cycles: gap $O(k \log k)$;
- $\mathcal{M}($$r)$: gap $O(k^2 r^3 \operatorname{polylog} kr)$ and $O(k^4 r^2 \operatorname{polylog} kr)$;

Known:

- cycles: gap $O(k \log k)$;
- $\mathcal{M}($$r)$: gap $O(k^2 r^3 \operatorname{polylog} kr)$ and $O(k^4 r^2 \operatorname{polylog} kr)$;
- non-planar graphs does not have the edge-EP property;

Known:

- cycles: gap $O(k \log k)$;
- $\mathcal{M}($$r)$: gap $O(k^2 r^3 \text{ polylog } kr)$ and $O(k^4 r^2 \text{ polylog } kr)$;
- non-planar graphs does not have the edge-EP property;
- lower bound for gap in non-acyclic graphs: $\Omega(k \log k)$.

Known:

- cycles: gap $O(k \log k)$;
- $\mathcal{M}($$)$: gap $O(k^2 r^3 \text{ polylog } kr)$ and $O(k^4 r^2 \text{ polylog } kr)$;
- non-planar graphs does not have the edge-EP property;
- lower bound for gap in non-acyclic graphs: $\Omega(k \log k)$.

Open:

- what other planar graphs have the edge-Erdős-Pósa property? (all planar graphs?)

Known:

- cycles: gap $O(k \log k)$;
- $\mathcal{M}($$r)$: gap $O(k^2 r^3 \operatorname{polylog} kr)$ and $O(k^4 r^2 \operatorname{polylog} kr)$;
- non-planar graphs does not have the edge-EP property;
- lower bound for gap in non-acyclic graphs: $\Omega(k \log k)$.

Open:

- what other planar graphs have the edge-Erdős-Pósa property? (all planar graphs?)
- better gap for r?

Thank you!