Induced minors and well-quasi-orderings

Jarosław Błasiok, Marcin Kamiński,
Jean-Florent Raymond and Théophile Trunck

20/02/2014

UNIWERSYTET WARSZAWSKI

ENS DE LYON

Definitions

- (partial) order: **reflexivity** + **antisymmetry** + **transitivity**;

Definitions

- (partial) order: **reflexivity** + **antisymmetry** + **transitivity**;
  - $(\mathbb{N}, \leqslant)$;
  - $(\mathcal{P}(\mathbb{N}), \subseteq)$;
  - most containment relations on graphs.

Definitions

- (partial) order: **reflexivity** + **antisymmetry** + **transitivity**;
  - $(\mathbb{N}, \leqslant)$;
  - $(\mathcal{P}(\mathbb{N}), \subseteq)$;
  - most containment relations on graphs.
- antichain: sequence of **non-comparable** elements;

Definitions

- (partial) order: **reflexivity** + **antisymmetry** + **transitivity**;
    - $(\mathbb{N}, \leqslant)$;
    - $(\mathcal{P}(\mathbb{N}), \subseteq)$;
    - most containment relations on graphs.
- antichain: sequence of **non-comparable** elements;
    - $\{0\}, \{1\}, \{2\}, \{3\}, \dots;$
    - $\dots$

Definitions

- (partial) order: **reflexivity** + **antisymmetry** + **transitivity**;
  - $(\mathbb{N}, \leqslant)$;
  - $(\mathcal{P}(\mathbb{N}), \subseteq)$;
  - most containment relations on graphs.
- antichain: sequence of **non-comparable** elements;
  - $\{0\}, \{1\}, \{2\}, \{3\}, \dots;$
  - $\dots$
- well-quasi-order (wqo): partial order **without infinite antichain** nor infinite decreasing sequence;

Definitions

- (partial) order: **reflexivity** + **antisymmetry** + **transitivity**;
  - $(\mathbb{N}, \leqslant)$;
  - $(\mathcal{P}(\mathbb{N}), \subseteq)$;
  - most containment relations on graphs.
- antichain: sequence of **non-comparable** elements;
  - $\{0\}, \{1\}, \{2\}, \{3\}, \dots;$
  - $\dots$
- well-quasi-order (wqo): partial order **without infinite antichain** nor infinite decreasing sequence;
  - every total order $(\mathbb{N}, \leqslant)$;
  - $(\mathcal{P}(\mathbb{N}), \subseteq)$ is not a wqo;
  - subsequence relation on sequences over a finite alphabet;
  - some containment relations on graphs.

Closed classes and obstruction sets

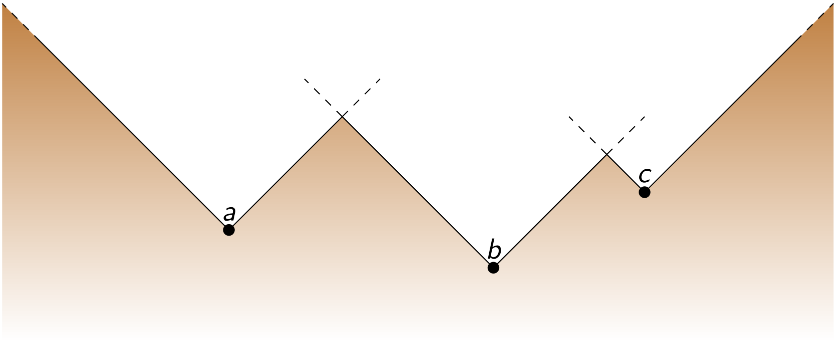
$(S, \leqslant)$ a partial order

- $P \subseteq S$ is a $\leqslant$-closed class if $x \in P$ whenever $x \leqslant y$ and $y \in P$.
  - planar graphs for $\leqslant_{\mathrm{m}}$, **tw** $< k$ for $\leqslant_{\mathrm{m}}$, $\gamma < k$ for $\leqslant_c$

Closed classes and obstruction sets

$(S, \leqslant)$ a partial order

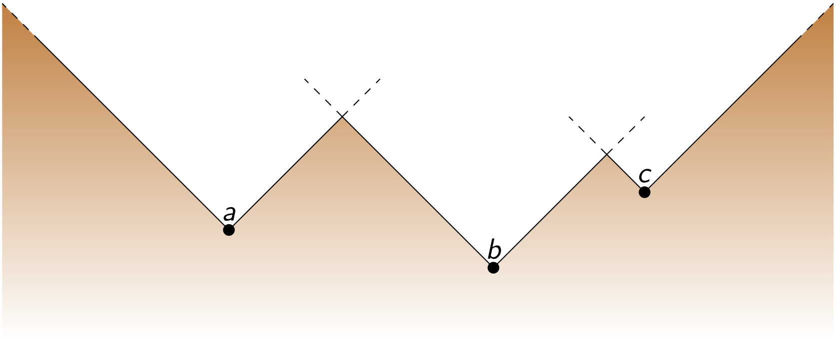
- $P \subseteq S$ is a $\leqslant$-closed class if $x \in P$ whenever $x \leqslant y$ and $y \in P$.
  - planar graphs for $\leqslant_{\mathrm{m}}$, **tw** $< k$ for $\leqslant_{\mathrm{m}}$, $\gamma < k$ for $\leqslant_c$



Closed classes and obstruction sets

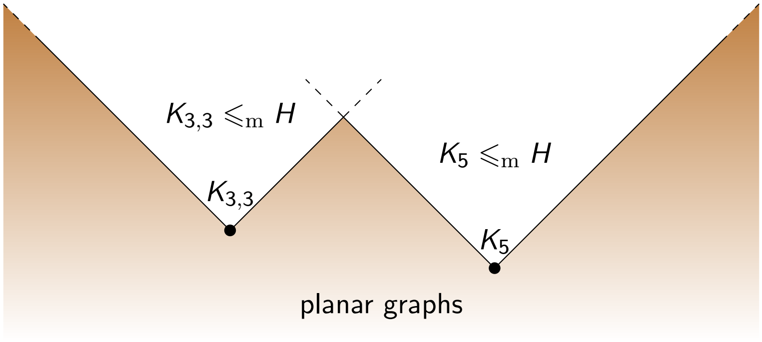
$(S, \leqslant)$ a partial order

- $P \subseteq S$ is a $\leqslant$-closed class if $x \in P$ whenever $x \leqslant y$ and $y \in P$.
  - planar graphs for $\leqslant_{\mathrm{m}}$, **tw** $< k$ for $\leqslant_{\mathrm{m}}$, $\gamma < k$ for $\leqslant_c$



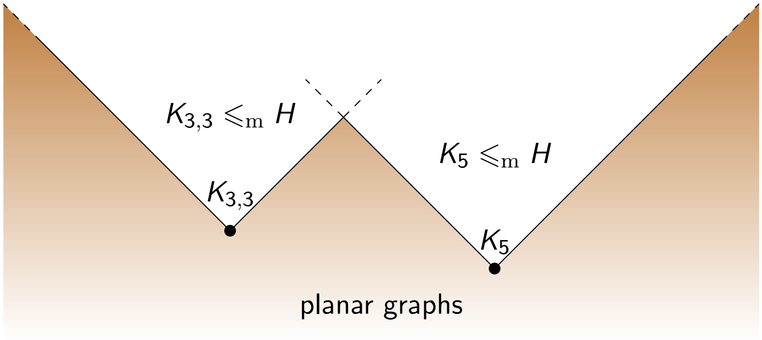
- $X \subseteq S$ obstruction set of P: $x \notin P \Leftrightarrow \exists y \in X, y \leqslant x$
  (characterization by forbidden substructure)

Finite obstruction sets



Key property of wqos: obstruction sets of $\leqslant$-closed properties are **finite**.

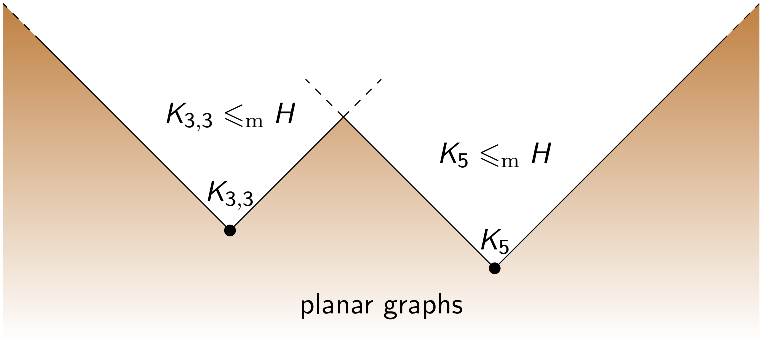
Finite obstruction sets



Key property of wqos: obstruction sets of $\leqslant$-closed properties are **finite**.

Consequence: checking $x \leqslant y$ efficiently $\Rightarrow$ deciding P efficiently.

Finite obstruction sets



Key property of wqos: obstruction sets of $\leqslant$-closed properties are **finite**.

Consequence: checking $x \leqslant y$ efficiently $\Rightarrow$ deciding P efficiently.

$\rightarrow$ **Goal:** prove that some containment relations are wqos.

Properties of wqos (1)

(reminder: wqo $\equiv$ no infinite sequence of non-comparable elements).

Remark (subset of a wqo)

If $(S, \leqslant)$ is a wqo and $U \subseteq S$ then $(U, \leqslant)$ is a wqo.

Remark (union of wqos)

If $(A, \leqslant)$ and $(B, \leqslant)$ are wqos, then $(A \cup B, \leqslant)$ is a wqo.

Properties of wqos (2)

Remark (weaker wqo)

If $(S, \leqslant)$ is a wqo and $\forall x, y \in S,\ x \leqslant y \Rightarrow x \leqslant' y$ then $(S, \leqslant')$ is a wqo.

Consequence: any antichain in $(S, \leqslant')$ is an antichain in $(S, \leqslant)$.

Remark (monotonicity and wqos)

If $(A, \leqslant_A)$ is a wqo and $\varphi\colon (A, \leqslant_A) \to (B, \leqslant_B)$ is surjective and monotone then $(B, \leqslant_B)$ is a wqo.

Tools

Theorem (Higman '52)

Let $(S, \leqslant)$ be a WQO. Finite sequences over S are WQO by $\leqslant^\star$.

Tools

Theorem (Higman '52)

Let $(S, \leqslant)$ *be a WQO. Finite sequences over* S *are WQO by* $\leqslant^\star$.

Theorem (Kruskal '60, Nash-Williams '63)

Trees are well-quasi-ordered by $\leqslant_{\mathrm{m}}$.

Tools

Theorem (Higman '52)

Let $(S, \leqslant)$ be a WQO. Finite sequences over S are WQO by $\leqslant^\star$.

Theorem (Kruskal '60, Nash-Williams '63)

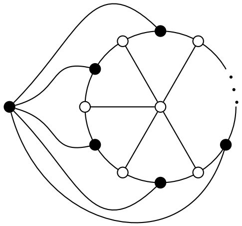
Trees are well-quasi-ordered by $\leqslant_{\mathrm{m}}$.

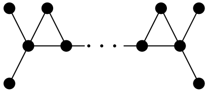
Theorem

Taking care of 2-connected graphs with labels is enough.

Containment relations

- Positive results: $(\mathcal{G}, \leqslant_{\mathrm{m}})$ [GMXX] (Wagner's conjecture), $(\mathcal{G}, \leqslant_{\mathrm{w.imm}})$ [GMXXIII] (Nash-Williams' Conjecture);
- Negative results: (induced) subgraph, contraction, induced (topological) minor, topological minor;
- Open problems: induced immersion and strong immersion.





Looking at subclasses

Defined in terms of graph invariants...

Theorem (Fellows, Hermelin, Rosamond '09)

- *Graphs of bounded feedback-vertex-set are wqo by topological minors;*
- *graphs with bounded vertex-cover are wqo by subgraphs;*
- *graphs with bounded circumference are wqo by induced minors.*

Looking at subclasses

Defined in terms of graph invariants...

Theorem (Fellows, Hermelin, Rosamond '09)

- *Graphs of bounded feedback-vertex-set are wqo by topological minors;*
- *graphs with bounded vertex-cover are wqo by subgraphs;*
- *graphs with bounded circumference are wqo by induced minors.*

...or in terms of excluded substructure.

Theorem (Thomas '85)

Graphs without K_4 as induced minor are wqo by induced minors.

Looking at subclasses

Defined in terms of graph invariants...

Theorem (Fellows, Hermelin, Rosamond '09)

- *Graphs of bounded feedback-vertex-set are wqo by topological minors;*
- *graphs with bounded vertex-cover are wqo by subgraphs;*
- *graphs with bounded circumference are wqo by induced minors.*

...or in terms of excluded substructure.

Theorem (Thomas '85)

Graphs without K_4 as induced minor are wqo by induced minors.

Theorem (Liu and Thomas '13+)

Graphs without  *as topo minor are wqo by topo minors.*

Dichotomy theorems

(reminder: wqo $\equiv$ no infinite sequence of non-comparable elements).

For a relation $\leqslant$, characterize graphs H such that $(\mathrm{Excl}_{\leqslant}(H), \leqslant)$ is a wqo.

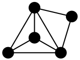
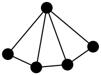
- [Damaschke, '90]: dichotomy theorem for induced subgraphs;
- [Ding, '92]: dichotomy theorem for subgraphs.

Dichotomy theorems

(reminder: wqo $\equiv$ no infinite sequence of non-comparable elements).

For a relation $\leqslant$, characterize graphs H such that $(\mathrm{Excl}_{\leqslant}(H), \leqslant)$ is a wqo.

- [Damaschke, '90]: dichotomy theorem for induced subgraphs;
- [Ding, '92]: dichotomy theorem for subgraphs.

$H_3 =$  $\mathrm{Gem} =$ 

Theorem (Błasiok, Kamiński, R., Trunck '14+)

The class of H-induced-minor-free graphs is wqo by $\leqslant_{\mathrm{im}}$ iff H is induced minor of Gem or H_3.

Sketch of the proof (1)

Direction $(\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H), \leqslant_{\mathrm{im}})$ wqo $\Rightarrow$ ($H \leqslant_{\mathrm{im}} \mathrm{Gem}$ or $H \leqslant_{\mathrm{im}} H_3$).

Remark

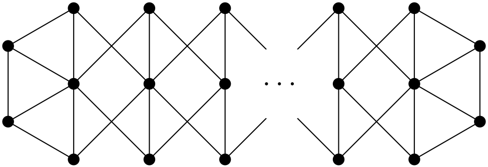
If $\mathcal{S}$ is an infinite antichain, H must be ind. minor of most of its terms.

Sketch of the proof (1)

Direction $(\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(H), \leqslant_{\mathrm{im}})$ wqo $\Rightarrow$ $(H \leqslant_{\mathrm{im}} \mathrm{Gem}$ or $H \leqslant_{\mathrm{im}} H_3)$.

Remark

If $\mathcal{S}$ is an infinite antichain, H must be ind. minor of most of its terms.



$\not\leqslant_{\mathrm{im}} K_5, K_5^-$

[Matoušek, Nešetřil, Thomas '88]

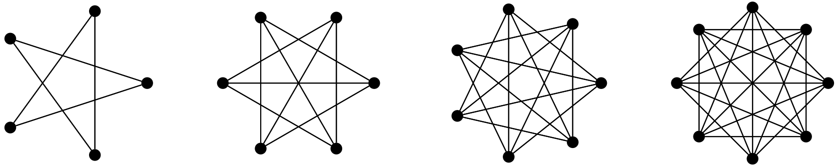
- $H \not\leqslant_{\mathrm{im}} K_5, K_5^- \quad \rightsquigarrow \quad \bar{H} \not\leqslant_{\mathrm{m}} 5 \cdot K_1$ and $\bar{H} \not\leqslant_{\mathrm{m}} 3 \cdot K_1 + K_2$;

Sketch of the proof (1)

Direction $(\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(H), \leqslant_{\mathrm{im}})$ wqo $\Rightarrow$ ($H \leqslant_{\mathrm{im}}$ Gem or $H \leqslant_{\mathrm{im}} H_3$).

Remark

If $\mathcal{S}$ is an infinite antichain, H must be ind. minor of most of its terms.



Antiholes

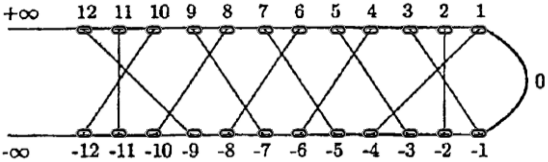
- $H \not\leqslant_{\mathrm{im}} K_5, K_5^-$ $\rightsquigarrow$ $\bar{H} \not\leqslant_{\mathrm{m}} 5 \cdot K_1$ and $\bar{H} \not\leqslant_{\mathrm{m}} 3 \cdot K_1 + K_2$;
- $\bar{H}$ is a linear forest;

Sketch of the proof (1)

Direction $(\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(H), \leqslant_{\mathrm{im}})$ wqo $\Rightarrow$ ($H \leqslant_{\mathrm{im}} \mathrm{Gem}$ or $H \leqslant_{\mathrm{im}} H_3$).

Remark

If $\mathcal{S}$ is an infinite antichain, H must be ind. minor of most of its terms.



[Ding '98]

- $H \not\leqslant_{\mathrm{im}} K_5, K_5^-$ $\rightsquigarrow$ $\bar{H} \not\leqslant_{\mathrm{m}} 5 \cdot K_1$ and $\bar{H} \not\leqslant_{\mathrm{m}} 3 \cdot K_1 + K_2$;
- $\bar{H}$ is a linear forest;
- $C_4 \not\leqslant_{\mathrm{m}} H$ $\rightsquigarrow$ $\bar{H} \not\leqslant_{\mathrm{m}} 2 \cdot P_2$.

Sketch of the proof (2)

- $\bar{H} \not\leq_{\mathrm{m}} 5 \cdot K_1$ and $\bar{H} \not\leq_{\mathrm{m}} 3 \cdot K_1 + K_2$;
- $\bar{H}$ is a linear forest;
- $\bar{H} \not\leq_{\mathrm{m}} 2 \cdot P_2$;

Sketch of the proof (2)

- $\bar{H} \not\leqslant_{\mathrm{m}} 5 \cdot K_1$ and $\bar{H} \not\leqslant_{\mathrm{m}} 3 \cdot K_1 + K_2$;
- $\bar{H}$ is a linear forest;
- $\bar{H} \not\leqslant_{\mathrm{m}} 2 \cdot P_2$;
- if $H_1 \leqslant_{\mathrm{im}} H_2$, $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_2)$ wqo $\Rightarrow$ $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_1)$ wqo.

Sketch of the proof (2)

- $\bar{H} \not\leqslant_{\mathrm{m}} 5 \cdot K_1$ and $\bar{H} \not\leqslant_{\mathrm{m}} 3 \cdot K_1 + K_2$;
- $\bar{H}$ is a linear forest;
- $\bar{H} \not\leqslant_{\mathrm{m}} 2 \cdot P_2$;
- if $H_1 \leqslant_{\mathrm{im}} H_2$, $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_2)$ wqo $\Rightarrow$ $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_1)$ wqo.

Check all small cases for $\bar{H}$ ($\leqslant 4$ connected components and $\leqslant 5$ vertices):

Sketch of the proof (2)

- $\bar{H} \not\leqslant_{\mathrm{m}} 5 \cdot K_1$ and $\bar{H} \not\leqslant_{\mathrm{m}} 3 \cdot K_1 + K_2$;
- $\bar{H}$ is a linear forest;
- $\bar{H} \not\leqslant_{\mathrm{m}} 2 \cdot P_2$;
- if $H_1 \leqslant_{\mathrm{im}} H_2$, $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_2)$ wqo $\Rightarrow$ $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_1)$ wqo.

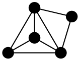
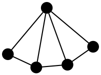
Check all small cases for $\bar{H}$ ($\leqslant 4$ connected components and $\leqslant 5$ vertices): all possible H are induced minors of Gem or H_3.

Sketch of the proof (3)

Direction ($H \leqslant_{\mathrm{im}} \mathrm{Gem}$ or $H \leqslant_{\mathrm{im}} H_3$) $\Rightarrow$ ($\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H), \leqslant_{\mathrm{im}}$) wqo.

Sketch of the proof (3)

Direction ($H \leqslant_{\mathrm{im}} \mathrm{Gem}$ or $H \leqslant_{\mathrm{im}} H_3$) $\Rightarrow$ ($\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(H), \leqslant_{\mathrm{im}}$) wqo.

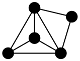
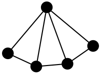
$H_3 =$  $\mathrm{Gem} =$ 

Theorem (Błasiok, Kamiński, R., Trunck '14+)

- ($\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(H_3), \leqslant_{\mathrm{im}}$) *is a wqo;*
- ($\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(Gem), \leqslant_{\mathrm{im}}$) *is a wqo.*

Sketch of the proof (3)

Direction ($H \leqslant_{\mathrm{im}} \mathrm{Gem}$ or $H \leqslant_{\mathrm{im}} H_3$) $\Rightarrow$ $(\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(H), \leqslant_{\mathrm{im}})$ wqo.

$H_3 =$  $\mathrm{Gem} =$ 

Theorem (Błasiok, Kamiński, R., Trunck '14+)

- $(\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(H_3), \leqslant_{\mathrm{im}})$ *is a wqo;*
- $(\mathrm{Excl}_{\leqslant_{\mathrm{im}}}(Gem), \leqslant_{\mathrm{im}})$ *is a wqo.*

Proof?

Proving wqo

How to prove wqo?

(1) Decomposition theorem $\rightarrow$ structure;

(2) Encode each graph G as a string s_G;

(3) Show that $s_H \preceq s_G \Rightarrow H \leqslant_{\mathrm{im}} G$;

(4) Prove that strings are wqo ($\sim$ Higman);

(5) By (3), graphs are wqo.

Decomposition

Theorem (Błasiok, Kamiński, R., Trunck '14+)

Decomposition theorem for graphs of $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_3) \setminus \mathsf{Excl}_{\leqslant_{\mathrm{im}}}(K_4)$*: cycle + complete multipartite.*

Theorem (Błasiok, Kamiński, R., Trunck '14+)

Decomposition theorem for graphs of $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(Gem) \setminus \mathsf{Excl}_{\leqslant_{\mathrm{im}}}(K_4)$*: tree-like structure.*

Blackboard.

The end

Theorem (Błasiok, Kamiński, R., Trunck '14+)

The class of H-induced-minor-free graphs is wqo by $\leqslant_{\mathrm{im}}$ iff H is induced minor of Gem or H_3.

Theorem (Błasiok, Kamiński, R., Trunck '14+)

- $(\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_3), \leqslant_{\mathrm{im}})$ *is a wqo;*
- $(\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(Gem), \leqslant_{\mathrm{im}})$ *is a wqo.*

Theorem (Błasiok, Kamiński, R., Trunck '14+)

Decomposition theorem for graphs of:

- $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(H_3) \setminus \mathsf{Excl}_{\leqslant_{\mathrm{im}}}(K_4)$: *cycle + multipartite;*
- $\mathsf{Excl}_{\leqslant_{\mathrm{im}}}(Gem) \setminus \mathsf{Excl}_{\leqslant_{\mathrm{im}}}(K_4)$: *tree-like structure.*

The end

Further research:

- dichotomy for contractions (✓);
- canonical antichains;
- induced immersions and strong immersions (hard).

Thank you!