An *edge* variant of Erdős–Pósa property

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December 2013

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Theorem (EP, classic version)

Every graph has

- **k** *vertex-disjoint cycles;*
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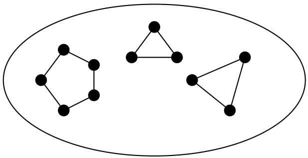

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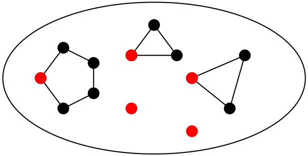

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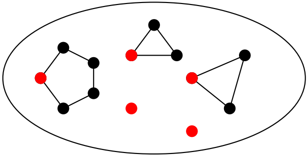

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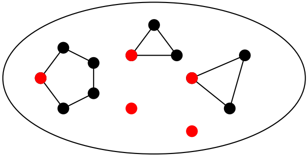

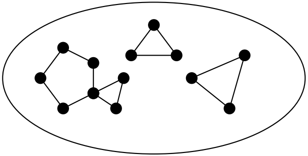

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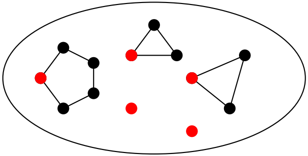

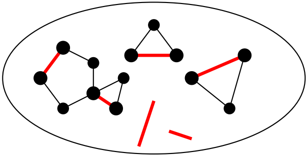

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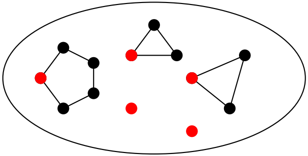

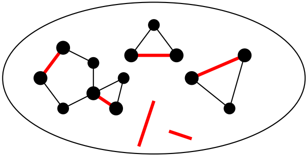

(relation packing / cover)

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Definition

$\mathcal{F}$ has the **Erdős–Pósa property** if $\exists f$ (*gap*) s.t. G has

- a (vertex-disjoint) packing of $\mathcal{F}$ of size k;
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(in this talk, $\mathcal{F} = \mathcal{M}(H)$)

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- **[Chekury, Chuzhoy 13]** $f_H(k) = O(k \text{ polylog} k)$.

(non-exhaustive list)

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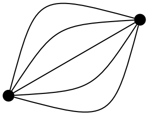
A simple remark

θ_r: multigraph with 2 vertices and r edges.



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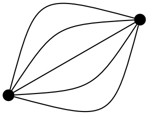

Remark

G has a cycle $\Leftrightarrow$ G contains a θ_2-model

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What about θ_r instead of θ_2?

Erdős–Pósa property for θ_r

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Almost nothing is known about this edge version.

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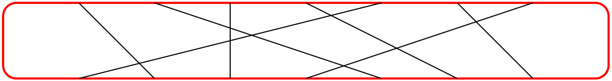

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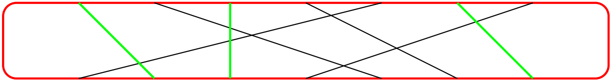

4. select only rk *parallel* edges.

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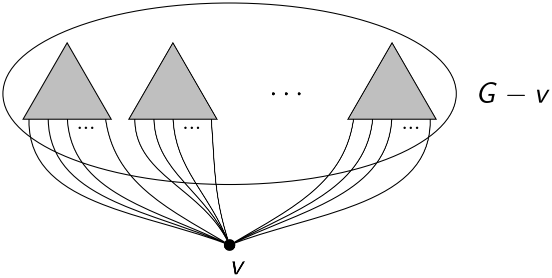

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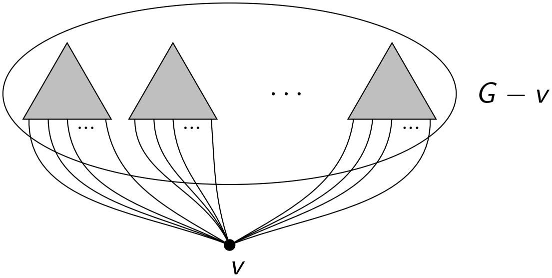

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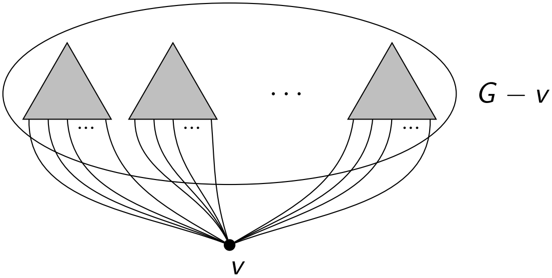
- split T into k connected edge-disjoint trees each having r neighbors of v.



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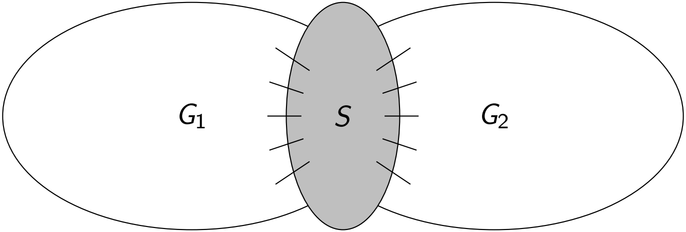

$S \supseteq$ vertices of θ_r-models in G_1 **and** G_2

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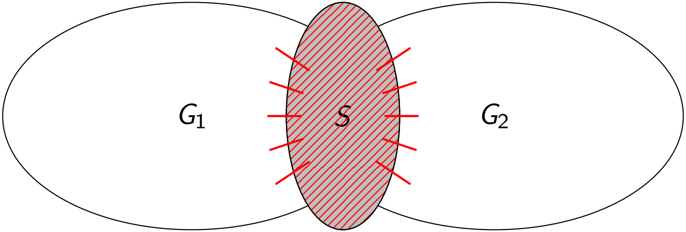

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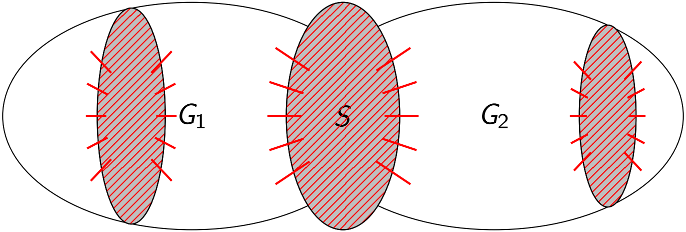

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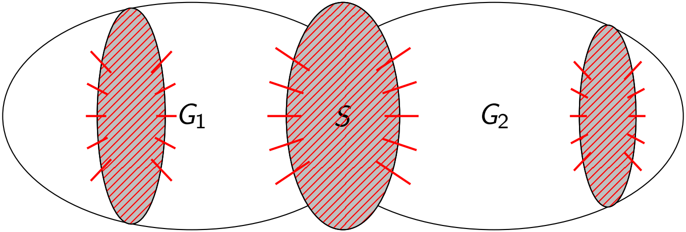


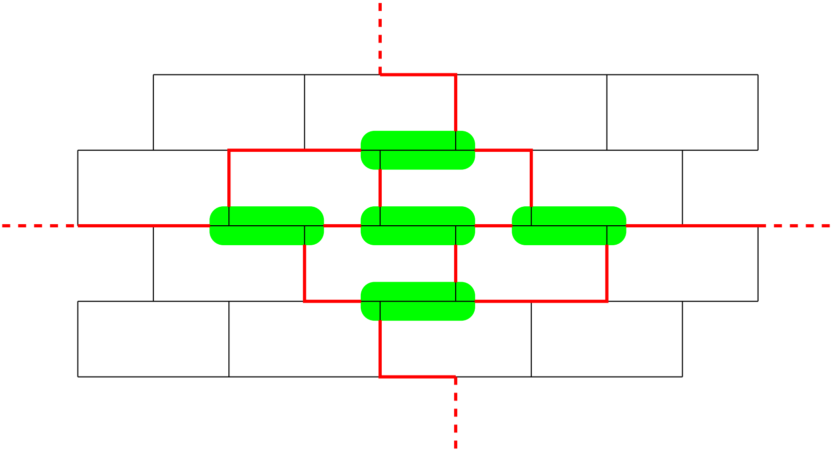
...

Eventually...

- First case: vertex-disjoint packing of θ_r of size k;
- Second case: edge-disjoint packing of θ_r of size k;
- Third case: edge-hitting set of size $O(k^3 r^3)$.

A last remark

$\mathcal{M}(H)$ has the edge-Erdős–Pósa property if H is planar.



One K_5-model but $\Omega(\sqrt{n})$ edges needed to destroy it...

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http://arxiv.org/abs/1311.1108

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