Polynomial gap extensions of the Erdős–Pósa theorem

Jean-Florent Raymond   Dimitrios M. Thilikos

LIRMM, Montpellier, France

09/09/2013

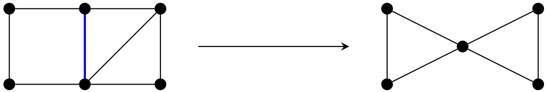
Laboratoire
Informatique
Robotique
Microélectronique
Montpellier

Minors and models

G can be **contracted** into $H \Leftrightarrow G$ is a H-**model** $\Leftrightarrow G \in \mathcal{M}(H)$

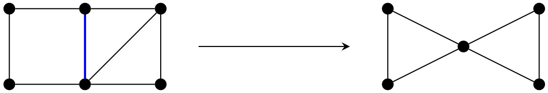
Minors and models

G can be **contracted** into $H \Leftrightarrow G$ is a H-**model** $\Leftrightarrow G \in \mathcal{M}(H)$



Minors and models

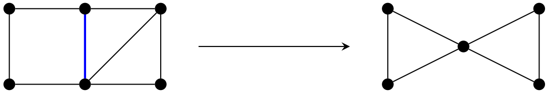
G can be **contracted** into $H \Leftrightarrow G$ is a H-**model** $\Leftrightarrow G \in \mathcal{M}(H)$



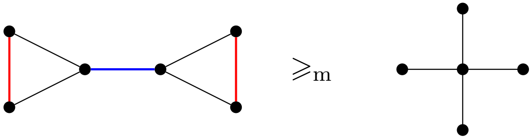
H is a **minor** of $G \Leftrightarrow G$ has a H-model as subgraph $\Leftrightarrow H \leqslant_{\mathrm{m}} G$

Minors and models

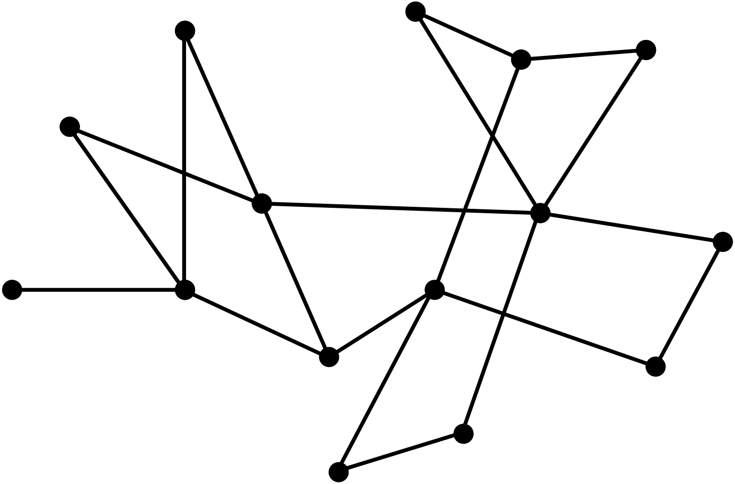
G can be **contracted** into $H \Leftrightarrow G$ is a H-**model** $\Leftrightarrow G \in \mathcal{M}(H)$



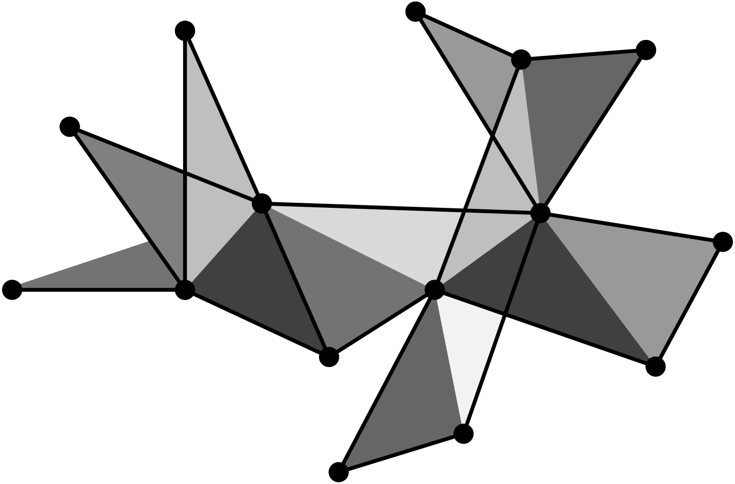
H is a **minor** of $G \Leftrightarrow G$ has a H-model as subgraph $\Leftrightarrow H \leqslant_{\mathrm{m}} G$



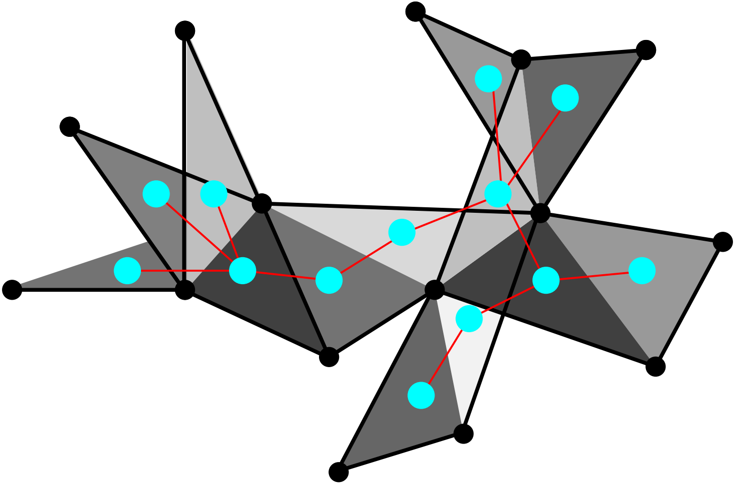
Tree decomposition and treewidth



Tree decomposition and treewidth



Tree decomposition and treewidth



Packing and covering $\mathcal{M}(H)$

H, G: graphs, S set of graphs

Packing of S in G: disjoint subgraphs of G that are in S

Packing and covering $\mathcal{M}(H)$

H, G: graphs, S set of graphs

Packing of S **in** G: disjoint subgraphs of G that are in S

$\rightarrow$ $\mathbf{pack}_{\mathcal{M}(H)}(G) =$ max number of disjoint H-models in G.

(**packing number**)

Packing and covering $\mathcal{M}(H)$

H, G: graphs, S set of graphs

Packing of S **in** G: disjoint subgraphs of G that are in S
$\rightarrow$ $\mathbf{pack}_{\mathcal{M}(H)}(G)$ = max number of disjoint H-models in G.
(**packing number**)

Cover of S **in** G: subset $X \subseteq V(G)$ s.t. $G \setminus X$ has no subgraph in S.

Packing and covering $\mathcal{M}(H)$

H, G: graphs, S set of graphs

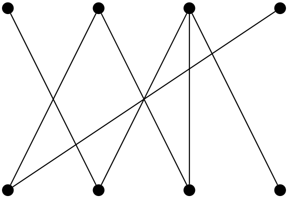
Packing of S **in** G: disjoint subgraphs of G that are in S
$\rightarrow$ $\textbf{pack}_{\mathcal{M}(H)}(G)$ = max number of disjoint H-models in G.
(**packing number**)

Cover of S **in** G: subset $X \subseteq V(G)$ s.t. $G \setminus X$ has no subgraph in S.
$\rightarrow$ $\textbf{cover}_{\mathcal{M}(H)}(G)$ = min size of a cover of $\mathcal{M}(H)$ in G.
(**covering number**)

Example: K_2 in bipartite graphs

K_2 maximum packing = maximum matching

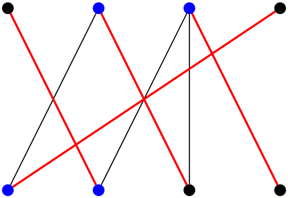
K_2 minimum cover = minimum vertex cover



Example: K_2 in bipartite graphs

K_2 maximum packing = maximum matching

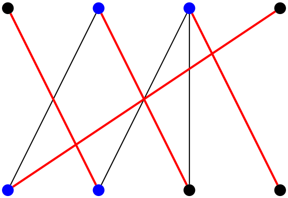
K_2 minimum cover = minimum vertex cover



Example: K_2 in bipartite graphs

K_2 maximum packing = maximum matching

K_2 minimum cover = minimum vertex cover

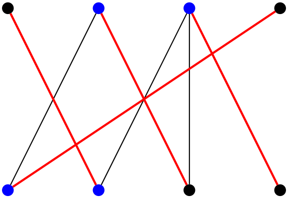


Relation between $\mathbf{pack}_{K_2}(G)$ and $\mathbf{cover}_{K_2}(G)$?

Example: K_2 in bipartite graphs

K_2 maximum packing = maximum matching

K_2 minimum cover = minimum vertex cover



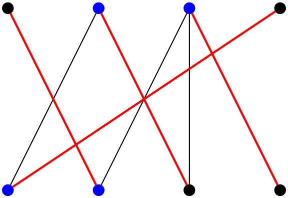
Relation between $\mathbf{pack}_{K_2}(G)$ and $\mathbf{cover}_{K_2}(G)$?

$$\mathbf{pack}_{K_2}(G) \leqslant \mathbf{cover}_{K_2}(G)$$

Example: K_2 in bipartite graphs

K_2 maximum packing = maximum matching

K_2 minimum cover = minimum vertex cover



Relation between $\mathbf{pack}_{K_2}(G)$ and $\mathbf{cover}_{K_2}(G)$?

$$\mathbf{pack}_{K_2}(G) = \mathbf{cover}_{K_2}(G) \qquad \text{(König's Theorem)}$$

And for other graphs?

In general,

is there a relation between $\mathbf{pack}_{\mathcal{M}(H)}(G)$ and $\mathbf{cover}_{\mathcal{M}(H)}(G)$?

And for other graphs?

In general,

is there a relation between $\textbf{pack}_{\mathcal{M}(H)}(G)$ and $\textbf{cover}_{\mathcal{M}(H)}(G)$?

$$\textbf{pack}_{\mathcal{M}(H)}(G) \leqslant \textbf{cover}_{\mathcal{M}(H)}(G)$$

And for other graphs?

In general,

is there a relation between $\mathbf{pack}_{\mathcal{M}(H)}(G)$ and $\mathbf{cover}_{\mathcal{M}(H)}(G)$?

$$\mathbf{pack}_{\mathcal{M}(H)}(G) \leqslant \mathbf{cover}_{\mathcal{M}(H)}(G) \overset{?}{\leqslant} f(\mathbf{pack}_{\mathcal{M}(H)}(G))$$

And for other graphs?

In general,

is there a relation between $\mathbf{pack}_{\mathcal{M}(H)}(G)$ and $\mathbf{cover}_{\mathcal{M}(H)}(G)$?

$$\mathbf{pack}_{\mathcal{M}(H)}(G) \leqslant \mathbf{cover}_{\mathcal{M}(H)}(G) \overset{?}{\leqslant} f(\mathbf{pack}_{\mathcal{M}(H)}(G))$$

If so: $\mathcal{M}(H)$ has **Erdős–Pósa Property** with **gap** f.

And for other graphs?

In general,

is there a relation between $\mathbf{pack}_{\mathcal{M}(H)}(G)$ and $\mathbf{cover}_{\mathcal{M}(H)}(G)$?

$$\mathbf{pack}_{\mathcal{M}(H)}(G) \leqslant \mathbf{cover}_{\mathcal{M}(H)}(G) \overset{?}{\leqslant} f(\mathbf{pack}_{\mathcal{M}(H)}(G))$$

If so: $\mathcal{M}(H)$ has **Erdős–Pósa Property** with **gap** f.

(EP Property for short)

And for other graphs?

In general,

is there a relation between $\mathbf{pack}_{\mathcal{M}(H)}(G)$ and $\mathbf{cover}_{\mathcal{M}(H)}(G)$?

$$\mathbf{pack}_{\mathcal{M}(H)}(G) \leqslant \mathbf{cover}_{\mathcal{M}(H)}(G) \overset{?}{\leqslant} f(\mathbf{pack}_{\mathcal{M}(H)}(G))$$

If so: $\mathcal{M}(H)$ has **Erdős–Pósa Property** with **gap** f.

(EP Property for short)

Question

For which graph H does $\mathcal{M}(H)$ have the EP Property? With which gap?

Two first results

Theorem (Erdős, Pósa '65)

Every G has either k disjoint cycles, or a set of $O(k \log k)$ vertices whose removal in G gives an acyclic graph.

Two first results

Theorem (Erdős, Pósa '65)

Every G has either k disjoint cycles, or a set of $O(k \log k)$ vertices whose removal in G gives an acyclic graph.
(EP Property is true for cycles with gap $O(k \log k)$)

Two first results

Theorem (Erdős, Pósa '65)

Every G has either k disjoint cycles, or a set of $O(k \log k)$ vertices whose removal in G gives an acyclic graph.
(EP Property is true for cycles with gap $O(k \log k)$)

Theorem (Robertson, Seymour '86)

The models of H have the Erdős-Pósa Property **iff** *H is planar.*

Two first results

Theorem (Erdős, Pósa '65)

Every G has either k disjoint cycles, or a set of $O(k \log k)$ vertices whose removal in G gives an acyclic graph.
(EP Property is true for cycles with gap $O(k \log k)$)

Theorem (Robertson, Seymour '86)

The models of H have the Erdős-Pósa Property **iff** *H is planar.*

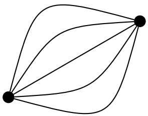
Question: polynomial gap for every planar graph?

Previous Work

Theorem (Fiorini, Joret, Sau '13)

$\mathcal{M}(\theta_r)$ has the Erdős–Pósa Property with gap $f_{\mathcal{M}(\theta_r)} = O(k \log k)$.

θ_5

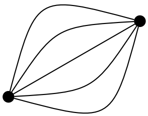


Previous Work

Theorem (Fiorini, Joret, Sau '13)

$\mathcal{M}(\theta_r)$ has the Erdős–Pósa Property with gap $f_{\mathcal{M}(\theta_r)} = O(k \log k)$.

θ_5



How: extending the proof of the Erdős–Pósa Theorem.

Our results

Theorem (R., Thilikos, '13)

For every H of pathwidth $\leqslant 2$, $\mathcal{M}(H)$ has the EP Property with gap

$$f_H(k) = 2^{O(|V(H)|^2)} k^2 \log k.$$

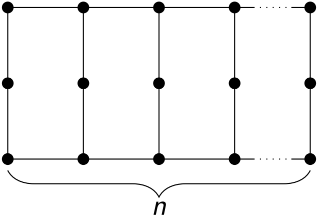
Our results

Theorem (R., Thilikos, '13)

For every H of pathwidth $\leqslant 2$, $\mathcal{M}(H)$ has the EP Property with gap

$$f_H(k) = 2^{O(|V(H)|^2)} k^2 \log k.$$

Graph of pathwidth $\leqslant 2$ on n vertices: minor of



How?

1. [Diestel, Jensen, Gorbunov, Thomassen, '99]
   large treewith ⇒ large *mesh* structure;

How?

1. [Diestel, Jensen, Gorbunov, Thomassen, '99]
   large treewith $\Rightarrow$ large *mesh* structure;
2. **Lemma:** every graph of **pw** $\leqslant 2$ is minor of a big enough mesh;

How?

1. [Diestel, Jensen, Gorbunov, Thomassen, '99]
   large treewith $\Rightarrow$ large *mesh* structure;
2. **Lemma:** every graph of **pw** $\leqslant 2$ is minor of a big enough mesh;
3. **Corollary:** large treewidth $\Rightarrow$ graph of **pw** $\leqslant 2$ as minor;

How?

1. [Diestel, Jensen, Gorbunov, Thomassen, '99]
   large treewith $\Rightarrow$ large *mesh* structure;
2. **Lemma:** every graph of **pw** $\leqslant 2$ is minor of a big enough mesh;
3. **Corollary:** large treewidth $\Rightarrow$ graph of **pw** $\leqslant 2$ as minor;
4. [Fomin, Saurabh, Thilikos '10]: translate this into a gap for EP Property.

$\rightsquigarrow$ EP for models of every graph of pathwidth $\leqslant 2$ with gap $O(k^2 \log k)$

Recent results

Theorem (Chekury, Chuzhoy, 2013)

H planar $\Rightarrow \mathcal{M}(H)$ *has the EP Property with a gap* $f(k) = k \operatorname{polylog} k$.

Recent results

Theorem (Chekury, Chuzhoy, 2013)

H planar $\Rightarrow \mathcal{M}(H)$ has the EP Property with a gap $f(k) = k \operatorname{polylog} k$.

Note: This new breakthrough result completely subsumes our results.

Conclusion and perspectives

- we proved that the gap of models of graphs of **pw** $\leqslant 2$ is

$$f(k) = O(k^2 \log k);$$

Conclusion and perspectives

- we proved that the gap of models of graphs of **pw** $\leqslant 2$ is

$$f(k) = O(k^2 \log k);$$

- [Chekury, Chuzhoy, 2013]: gap of models of planar graphs is

$$f(k) = k \text{ polylog } k.$$

Conclusion and perspectives

- we proved that the gap of models of graphs of **pw** $\leqslant 2$ is

$$f(k) = O(k^2 \log k);$$

- [Chekury, Chuzhoy, 2013]: gap of models of planar graphs is

$$f(k) = k \text{ polylog } k.$$

New questions:

1. can we prove a gap of $O(k \log k)$ for every planar?

Conclusion and perspectives

- we proved that the gap of models of graphs of **pw** $\leqslant 2$ is

$$f(k) = O(k^2 \log k);$$

- [Chekury, Chuzhoy, 2013]: gap of models of planar graphs is

$$f(k) = k \text{ polylog } k.$$

New questions:

1. can we prove a gap of $O(k \log k)$ for every planar?
2. what about variants of EP Property (e.g. for topological minors)?

Conclusion and perspectives

- we proved that the gap of models of graphs of **pw** $\leqslant 2$ is

$$f(k) = O(k^2 \log k);$$

- [Chekury, Chuzhoy, 2013]: gap of models of planar graphs is

$$f(k) = k \text{ polylog } k.$$

New questions:

1. can we prove a gap of $O(k \log k)$ for every planar?
2. what about variants of EP Property (e.g. for topological minors)?

Thank you!