Impulsive control of portfolios

Jan Palczewski* Łukasz Stettner†

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Abstract

In the paper a general model of a market with asset prices and economical factors of Markovian structure is considered. The problem is to find optimal portfolio strategies maximizing a discounted infinite horizon reward functional consisting of an integral term measuring quality of the portfolio at each moment and a discrete term measuring the reward from consumption. There are general transaction costs which, in particular, cover fixed plus proportional costs. It is shown, under general conditions, that there exists an optimal impulse strategy and the value function is a solution to the Bellman equation which corresponds to suitable quasi-variational inequalities.

Keywords: Markov process, impulsive control, Bellman equation, portfolio optimization, transaction costs

1. Introduction

On a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ consider a market modeled by a time homogeneous Markov process $\big(S(t), X(t)\big)_{t\geq 0}$, where $S(t) = \big(S^1(t), \ldots, S^d(t)\big) \in (0,\infty)^d$ denotes prices of d assets and $X(t) \in \mathbb{R}^m$ stands for m economic factors. The use of economic factors was fully justified in [2]. Firstly, it expands capabilities of the model to some extent. Secondly, it facilitates and improves the quality of statistical estimation of model parameters. We shall assume that $\big(S(t), X(t)\big)$ has right continuous and left limited trajectories and satisfies the so called Feller property, i.e. its semigroup transforms the space of continuous bounded functions vanishing at infinity into itself. We invest in assets. Let $N^i(t)$ be the number of shares of the i-th asset in our portfolio at time $t \geq 0$. The vector $N(t) = \big(N^1(t), \ldots, N^d(t)\big)$ describes the portfolio contents at time t. We shall assume that $N^i(t) \in [0,\infty)$, $i = 1, \ldots, d$, which means that neither short-selling nor borrowing is allowed. Let $C_k \in [0,\infty)$ be the amount of money withdrawn from the portfolio and consumed at time τ_k, $k = 1, 2, \ldots$. We are interested in maximization of the discounted infinite horizon reward functional

$$\mathbb{E}\Big\{ \int_0^\infty e^{-\alpha s} F\big(Y(s)\big) ds + \sum_{k=1}^\infty e^{-\alpha \tau_k} G(C_k) \Big\},$$

*Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, 00-950 Warszawa, Poland, (e-mail: jpalczew@mimuw.edu.pl). Research supported by grant KBN 1-P03A-012-28.

†Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, 00-950 Warszawa, Poland, (e-mail: stettner@impan.gov.pl)

where $Y(s) = \big(N(s), S(s), X(s)\big)$. We assume that changes of our portfolio strategy $N(t)$ are of impulsive form. An impulsive strategy $\Pi = \big((N_0, 0), (N_1, C_1, \tau_1), (N_2, C_2, \tau_2), \dots\big)$ is a sequence of non-decreasing stopping times τ_i, $\tau_0 = 0$, and $\mathcal{F}_{\tau_i}$-measurable random vectors $N_i \in [0, \infty)^d$, $C_i \in [0, \infty)$. At a random time τ_i the portfolio is changed from N_{i-1} to N_i and the consumption of C_i is executed. The choice of N_i and C_i should satisfy the self-financing property i.e.

$$N_i \cdot S(\tau_i) + C_i + c\big(N_{i-1}, N_i, S(\tau_i)\big) = N_{i-1} \cdot S(\tau_i), \tag{1}$$

where $N_i \cdot S(\tau_i)$ means the Euclidean scalar product of the vectors N_i and $S(\tau_i)$ and $c\big(N_{i-1}, N_i, S(\tau_i)\big)$ is a positive transaction cost given the assets' price $S(\tau_i)$. In the paper we will omit a dot in the notation for the scalar product of vectors unless it leads to ambiguity. We shall assume that the cost function $c : [0, \infty)^d \times [0, \infty)^d \times (0, \infty)^d \to \mathbb{R}$ is of the form $c(\eta_0, \eta_1, s) = K + \tilde{c}(\eta_0, \eta_1, s)$, where

i) $K > 0$,

ii) $\tilde{c}(\eta_0, \eta_1, s)$ is continuous,

iii) $\tilde{c}(\eta_0, \eta_1 + \gamma, s) - \tilde{c}(\eta_0, \eta_1, s) \leq \beta\, \gamma \cdot s$, for any $\gamma \in [0, \infty)^d$ and fixed $\beta > 0$,

iv) $\tilde{c}(\alpha\eta_0, \alpha\eta_1, s) \leq \alpha\tilde{c}(\eta_0, \eta_1, s)$ for $\alpha \geq 1$,

v) $\tilde{c}(\eta_0, \eta_1, s) \geq 0$,

vi) $\tilde{c}(\eta_0 + \gamma, \eta_1 + \gamma, s) \leq \tilde{c}(\eta_0, \eta_1, s)$ for $\gamma \in [0, \infty)^d$.

The continuity of the cost function (ii) guarantees that a small change of the transaction does not modify the cost significantly. The condition iii) sets an upper bound for the transaction costs: the cost of acquiring γ shares cannot exceed a certain multiple of the price paid. We do not impose $\beta < 1$. The cost (per share) of managing a portfolio goes down as a size of the portfolio increases on a real market – this is grasped in iv) and vi). We do not assume that the cost function c is subadditive, i.e. it may not satisfy the triangle inequality, thus it can be optimal to make multiple transactions at the same moment.

The above requirements are not restrictive. They cover the case of fixed plus proportional transaction costs, i.e.

$$c(n, n_1, s) = K + (n - n_1)^+ \alpha s + (n - n_1)^- \tilde{\alpha} s, \tag{2}$$

for diagonal matrices with non-negative entries $\alpha, \tilde{\alpha} \in \mathbb{R}^{d\times d}$. Here, n^+ means a positive part of every coordinate and n^- – a negative part of every coordinate. Moreover, the following cost functions fulfill our requirements for the cost function:

$$\begin{aligned} c(n, n_1, s) &= K + \tilde{c}(n, n_1, s), \\ c(n, n_1, s) &= \max\big(K, \tilde{c}(n, n_1, s)\big), \end{aligned}$$

where

$$\tilde{c}(n, n_1, s) = \sum_{i=1}^{d} \int_0^{(n-n_1)_i^+} z_i(t)dt + \sum_{i=1}^{d} \int_0^{(n-n_1)_i^-} \hat{z}_i(t)dt$$

and functions z_i, $\hat{z}_i$ are integrable and non-negative.

Given $y = (\eta, s, x) \in E := [0, \infty)^d \times (0, \infty)^d \times \mathbb{R}^m$ we shall denote by $\mathcal{A}(y)$ the class of impulsive strategies for the process $\big(S(t), X(t)\big)$ with the initial state $S(0) = s$, $X(0) = x$, given the initial portfolio $N(0) = \eta$. Observe that the set $\mathcal{A}(y)$ of admissible strategies depends strongly on the initial point (η, s, x). Condition (1) defines the set of possible transactions. In the case of the cost function (2), if the wealth of the portfolio is smaller than K no transaction can occur because there are no funds to pay for it. However, if $N_{i-1} \cdot S(\tau_i) \geq K$ the set of possible transactions is non-empty.

The form of transaction costs fully justifies the use of impulsive strategies, since we cannot trade continuously in the presence of a constant term in the cost function. One can see that there is a remarkable difference between classical optimal impulsive control problems (see [1]) and the one considered here. Namely, costs of the impulses do not appear as a penalizing term in the reward functional; they are encoded in the set of available controls. This type of problems is widely discussed in the literature. If we take $F \equiv 0$ and G – a utility function, we obtain a problem of optimal portfolio selection with consumption (see [6], [8], [11]). The integral part of the reward functional appears in various banking and cash management applications (see eg. third chapter of [14] or [4]). It measures in particular divergence of the portfolio from a selected benchmark (see [3], [15]), proper diversification (see [12]) or variance (see section 6c for more examples). Frequently function F depends on time, which is achieved in our model by extending the set of economic factors with a deterministic variable denoting time.

In the aforementioned papers authors consider models without economic factors and manage to prove existence of optimal control in special cases. Economic factors are dealt with in [2]. A common trait of the cited papers is the approach based on quasi-variational inequalities, introduced in [1]. In the paper a different technique is used to combine transaction costs with a general model of stock prices depending on economic factors. Existence of an optimal Markovian strategy under weak conditions imposed on transaction costs mechanism and price processes is then proved. Examples of models that satisfy above conditions are shown.

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2. Preliminary results

The following proposition is a very important result used throughout the paper. It shows a lower bound for the multiplicity of a trading strategy in which the savings on transaction costs are invested, for simplicity, in the asset 1. This result explains the meaning of the assumptions imposed on the cost function.

PROPOSITION 2.1. Let $\Pi = \big((N_0, \tau_0), (N_1, C_1, \tau_1), \dots\big) \in \mathcal{A}(N_0, s, x)$ be an admissible strategy and $\alpha > 1$. There exists a strategy $\tilde{\Pi} = \big((\tilde{N}_0, \tau_0), (\tilde{N}_1, C_1, \tau_1), \dots\big) \in \mathcal{A}(\alpha N_0, s, x)$ with $\tilde{N}_0 = \alpha N_0$ such that

$$\tilde{N}_i^1 = \alpha N_i^1 + \gamma_i,$$
$$\tilde{N}_i^k = \alpha N_i^k, \quad k = 2, \dots, d,$$

where

$$\gamma_i \geq \gamma_{i-1} + \frac{\frac{\alpha-1}{\beta+1} K}{S^1(\tau_i)},$$

and β is a constant from condition iii) in the definition of the cost function.

Proof. We start with an initial portfolio $\tilde{N}_0 = \alpha N_0$. We always keep at least αN_i stocks after i-th transaction. The structure of transaction costs ensures that we make savings bounded from below in wealth at each transaction. These, invested in S_1, cumulate. More precisely, γ_i denotes the cumulated savings in the number of stocks of S_1, $\gamma_0 = 0$. We construct $\tilde{\Pi}$ in such a way that

$$\tilde{N}_i - \gamma_i \epsilon_1 = \alpha N_i, \tag{3}$$

where $\epsilon_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$ and γ_i is an $\mathcal{F}_{\tau_i}$-measurable random variable.

The construction of the strategy $\tilde{\Pi}$ is by induction. At the beginning $\tilde{N}_0 = \alpha N_0$. Then $\tilde{N}_1$ is defined as a fixed point of

$$\tilde{N}_0 S(\tau_1) = \tilde{N}_1 S(\tau_1) + C_1 + c\big(\tilde{N}_0, \tilde{N}_1, S(\tau_1)\big).$$

satisfying (3). Apparently,

$$\alpha N_0 S(\tau_1) = (\alpha N_1 + \gamma_1 \epsilon_1) S(\tau_1) + C_1 + K + \tilde{c}\big(\alpha N_0, \alpha N_1 + \gamma_1 \epsilon_1, S(\tau_1)\big) = (\#).$$

By definition

$$\alpha N_0 S(\tau_1) = \alpha N_1 S(\tau_1) + \alpha C_1 + \alpha c\big(N_0, N_1, S(\tau_1)\big) \geq \alpha N_1 S(\tau_1) + C_1 + \alpha K + \tilde{c}\big(\alpha N_0, \alpha N_1, S(\tau_1)\big).$$

Therefore,

$$\begin{aligned}
(\#) &\leq (\alpha N_1 + \gamma_1 \epsilon_1) S(\tau_1) + C_1 + K + \tilde{c}\big(\alpha N_0, \alpha N_1, S(\tau_1)\big) + \beta \gamma_1 S^1(\tau_1) \\
&\leq \alpha N_1 S(\tau_1) + C_1 + \alpha K + \tilde{c}\big(\alpha N_0, \alpha N_1, S(\tau_1)\big) + \gamma_1 S^1(\tau_1) + (1-\alpha) K + \beta \gamma_1 S^1(\tau_1) \\
&\leq \alpha N_0 S(\tau_1) + \gamma_1 S^1(\tau_1) + (1-\alpha) K + \beta \gamma_1 S^1(\tau_1) \\
&= \tilde{N}_0 S(\tau_1) + (1-\alpha) K + (\beta + 1) \gamma_1 S^1(\tau_1).
\end{aligned}$$

So we obtain a lower bound for γ_1

$$(1-\alpha) K + (\beta+1) \gamma_1 S^1(\tau_1) \geq 0,$$

or equivalently

$$\gamma_1 \geq \frac{\frac{\alpha-1}{\beta+1} K}{S^1(\tau_1)}.$$

For the sake of clarity we present further steps of the induction. Let $\tilde{N}_i$ be defined as a fixed point of

$$\tilde{N}_{i-1} S(\tau_i) = \tilde{N}_i S(\tau_i) + C_i + c\big(\tilde{N}_{i-1}, \tilde{N}_i, S(\tau_i)\big)$$

satisfying (3). Apparently,

$$(\alpha N_{i-1} + \gamma_{i-1} \epsilon_1) S(\tau_i) = (\alpha N_i + \gamma_i \epsilon_1) S(\tau_i) + C_i + K + \tilde{c}\big(\alpha N_{i-1} + \gamma_{i-1} \epsilon_1, \alpha N_i + \gamma_i \epsilon_1, S(\tau_i)\big) = (\#\#).$$

By definition

$$\alpha N_{i-1} S(\tau_i) = \alpha N_i S(\tau_i) + \alpha C_i + \alpha c\big(N_{i-1}, N_i, S(\tau_i)\big) \geq \alpha N_i S(\tau_i) + C_i + \alpha K + \tilde{c}\big(\alpha N_{i-1}, \alpha N_i, S(\tau_i)\big).$$

Therefore,

$$\begin{aligned}(\#\#) &\leq (\alpha N_i + \gamma_i\epsilon_1)S(\tau_i) + C_i + K + \tilde{c}\big(\alpha N_{i-1} + \gamma_{i-1}\epsilon_1, \alpha N_i + \gamma_{i-1}\epsilon_1, S(\tau_i)\big)\\ &\quad + \beta(\gamma_i - \gamma_{i-1})S^1(\tau_i)\\ &\leq \alpha N_i S(\tau_i) + C_i + \alpha K + \tilde{c}\big(\alpha N_{i-1}, \alpha N_i, S(\tau_i)\big) + \gamma_i S^1(\tau_i) + (1-\alpha)K\\ &\quad + \beta(\gamma_i - \gamma_{i-1})S^1(\tau_i)\\ &\leq \alpha N_{i-1}S(\tau_i) + \gamma_{i-1}S^1(\tau_i) + (\gamma_i - \gamma_{i-1})S^1(\tau_i) + (1-\alpha)K + \beta(\gamma_i - \gamma_{i-1})S^1(\tau_i)\\ &= \tilde{N}_{i-1}S(\tau_i) + (1-\alpha)K + (\beta+1)(\gamma_i - \gamma_{i-1})S^1(\tau_i).\end{aligned}$$

Consequently, we obtain a lower bound for γ_i

$$(1-\alpha)K + (\beta+1)(\gamma_i - \gamma_{i-1})S^1(\tau_i) \geq 0,$$

or

$$\gamma_i - \gamma_{i-1} \geq \frac{\frac{\alpha-1}{\beta+1}K}{S^1(\tau_i)}.$$

■

Given an impulsive strategy $\Pi \in \mathcal{A}(y)$ we define the **share holding process** $N^{\Pi}(t)$

$$N^{\Pi}(t) = \sum_{i=0}^{\infty} 1_{[\tau_i, \tau_{i+1})}(t)N_i,$$

and the **wealth process** $W^{\Pi}(t) = N^{\Pi}(t) \cdot S(t)$. If two or more impulses occur at the same moment, only the last one plays a role in the share holding process and the wealth process, since $[a, a) = \emptyset$.

Throughout the paper we shall require the following non-explosiveness assumption.

Assumption F. The wealth process for any admissible impulsive strategy is finite on any compact interval $[0, T]$.

LEMMA 2.2. Under assumption F for any $\Pi \in \mathcal{A}(y)$, $y = (\eta, s, x) \in E$ and $L > 0$ we have

$$\mathbb{P}^{(s,x)}\big(\lim_{n\to\infty} \tau_n \leq L\big) = 0.$$

Proof. Fix an admissible strategy Π and $L > 0$. Assume that $\mathbb{P}^{(s,x)}\big(\lim_{n\to\infty} \tau_n \leq L\big) > 0$ and denote this event by A. All our further reasoning will be held on the set A. The strategy funds an infinite number of transactions. It means that it earns an infinite amount of money (although it may be spent on transaction costs). We shall construct a strategy whose wealth is infinite in some point in $[0, L]$ thus contradicting the assumption F.

Let us fix $\alpha > 1$. Let $\tilde{\Pi}$ be the strategy constructed in Proposition 2.1. The savings cumulated in S^1 are bounded by

$$\gamma_i - \gamma_{i-1} \geq \frac{\frac{\alpha-1}{\beta+1}K}{S^1(\tau_i)}.$$

Since S^1 has cádlág trajectories, which are bounded from above on $[0, L]$ for each $\omega \in \Omega$, γ_i tends to infinity. Hence, the wealth $\tilde{W}(t)$ of the portfolio $\tilde{\Pi}$ satisfies

$$\tilde{W}(\tau) \geq \lim_{i\to\infty} \gamma_i S^1(\tau),$$

which contradicts the assumption F. ∎

Notice that under assumption F each admissible strategy $\Pi = \big((N_0, 0), (N_1, C_1, \tau_1), \dots\big)$ satisfies

$$\lim_{i\to\infty} \tau_i = \infty, \quad \text{a.s.}$$

3. The optimization problem

For an admissible strategy Π for $(\eta, s, x) \in E$ let us introduce the state process

$$Y(t) = \begin{pmatrix} N^{\Pi}(t) \\ S(t) \\ X(t) \end{pmatrix} \in E.$$

To make the notation easier let us denote by $\mathbb{P}^y$ the probability measure $\mathbb{P}^{(s,x)}$, where $y = (\eta, s, x) \in E$.

Let $F : E \to \mathbb{R}_+$ be a reward function that measures the quality of the portfolio at each moment and $G : [0, \infty) \to \mathbb{R}_+$ the reward function that measures the quality of consumption. We assume only continuity of F and G, and the following

Assumption UF.

$$F(0, s, x) = \inf_{\eta\in[0,\infty)^d} F(\eta, s, x), \quad (s, x) \in (0, \infty)^d \times \mathbb{R}^m.$$

Our goal is to maximize the **reward functional**

$$J(\Pi) = \mathbb{E}^{(s,x)}\Big\{ \int_0^{\infty} e^{-\alpha t} F\big(Y^{\Pi}(t)\big) dt + \sum_{k=1}^{\infty} e^{-\alpha\tau_k^{\Pi}} G\big(C_k^{\Pi}\big)\Big\} \tag{4}$$

over the set of admissible strategies Π for $(\eta, s, x) \in E$. Let us introduce a **value function**

$$v(y) = \sup\{J(\Pi) : \quad \Pi \in \mathcal{A}(y)\}, \quad y \in E. \tag{5}$$

To formulate a Bellman equation we need to introduce a **switching operator**

$$Mu(\eta, s, x) = \sup\big\{G(\varsigma) + u(\eta_1, s, x) : \quad (\eta_1, \varsigma) \in \Gamma(\eta, s)\big\} \vee u(0, s, x),$$

where

$$\Gamma(\eta, s) = \big\{(\eta_1, \varsigma) \in [0, \infty)^d \times [0, \infty) : \quad \eta_1 \cdot s + \varsigma + c(\eta, \eta_1, s) = \eta \cdot s\big\}$$

and $\sup \emptyset = -\infty$.

DEFINITION 3.1. A measurable function $I = (I_1, I_2) : E \to [0, \infty)^d \times [0, \infty)$ is called an **impulse function** for Mu if

$$Mu(\eta, s, x) = G\big(I_2(\eta, s, x)\big) + u\big(I_1(\eta, s, x), s, x\big) \qquad \text{and} \qquad I(\eta, s, x) \in \Gamma(\eta, s).$$

Now we can define the **Bellman operator**

$$\mathcal{G}u(y) = \sup_{\tau} \mathbb{E}^{\,y}\Big(\int_0^{\tau} e^{-\alpha t} F\big(Y(t)\big) dt + e^{-\alpha \tau} Mu\big(Y(\tau)\big)\Big)$$

for any measurable function $u : E \to \mathbb{R}$ such that Mu is well-defined.

Let us denote by $C^{\mathcal{B}}(K;L)$ the set of bounded continuous functions acting from K into L, by $C^{\mathcal{L}}(K;L)$ the set of lower semi-continuous (l.s.c.) functions, and by $C^{\mathcal{U}}(K;L)$ is the set of upper semi-continuous (u.s.c.) functions.

LEMMA 3.2. M transforms $C(E,\mathbb{R})$ and $C^{\mathcal{U}}(E;\mathbb{R})$ into itself and if G is bounded $C^{\mathcal{B}}(E,\mathbb{R})$ into itself. Moreover, for any continuous, non-negative function u there exists an impulse function for Mu.

Proof. Denote by $\tilde{\Gamma}(\eta, s, x)$ the set of possible impulses:

$$\tilde{\Gamma}(\eta, s, x) = \big\{((\eta_1, s, x), \varsigma_1) \in E \times [0,\infty) : \quad (\eta_1, \varsigma_1) \in \Gamma(\eta, s)\big\} \cup \{((0, s, x), 0)\}.$$

Each set $\tilde{\Gamma}(\eta, s, x)$ is compact, since c is a continuous function. Take $u \in C^{\mathcal{U}}(E;\mathbb{R})$ and a sequence $y_n \to y$ in E. Denote by $(z_n, \varsigma_n) \in \tilde{\Gamma}(y_n)$ maximizers for y_n i.e. $Mu(y_n) = G(\varsigma_n) + u(z_n)$. Since $\bigcup_{n\in\mathbb{N}} \tilde{\Gamma}(y_n)$ is bounded, its closure is compact and we can find a subsequence of (z_n, ς_n) converging to some (z, ς) in $E \times [0,\infty)$. For simplicity such a subsequence we shall denote again by (z_n, ς_n). By u.s.c. of u we have $\limsup_{n\to\infty} u(z_n) \le u(z)$. By continuity of G we have $\lim_{n\to\infty} G(\varsigma_n) = G(\varsigma)$. Moreover, $(z, \varsigma) \in \tilde{\Gamma}(y)$ by continuity of c. Hence $G(\varsigma) + u(z) \le Mu(y)$ and Mu is u.s.c.

Now we shall prove that Mu is l.s.c. for a continuous function u. As above let us take a sequence $y_n \to y$ in E and fix $\epsilon > 0$. Denote by (z, ς) the maximizer of $Mu(y)$ i.e. $Mu(y) = G(\varsigma) + u(z)$. By continuity of u and G there exists $\delta > 0$ such that

$$\begin{aligned} \tilde{z} \in B_1 \quad &\Longrightarrow \quad |u(\tilde{z}) - u(z)| \le \epsilon, \\ \tilde{\varsigma} \in B_2 \quad &\Longrightarrow \quad |G(\tilde{\varsigma}) - G(\varsigma)| \le \epsilon, \end{aligned}$$

where $B_1 = \{\tilde{z} \in E : \quad \|z - \tilde{z}\| \le \delta\}$ and $B_2 = \{\tilde{\varsigma} \in [0,\infty) : \quad \|\varsigma - \tilde{\varsigma}\| \le \delta\}$. Moreover, $B_1 \times B_2 \cap \tilde{\Gamma}(y_n) \neq \emptyset$ for sufficiently large n. Hence $\liminf_{n\to\infty} Mu(y_n) \ge u(z) + G(\varsigma) - 2\epsilon$. Since ϵ can be taken arbitrarily small, $\liminf_{n\to\infty} Mu(y_n) \ge u(z) + G(\varsigma)$ and Mu is l.s.c. Consequently, Mu is continuous for a continuous u. Its boundedness is obvious.

The existence of an impulse function for Mv is a direct consequence of Proposition D.3 in [7]. ■

Let us denote

$$v_0(y) = \mathbb{E}^{\,y} \int_0^{\infty} e^{-\alpha t} F\big(Y(t)\big) dt, \quad y \in E,$$

and $v_{i+1} = \mathcal{G}v_i$, $i = 0, 1, \ldots$. Recall that v is a value function.

LEMMA 3.3. If $F, G \in C^{\mathcal{B}}(E, \mathbb{R}_+)$ and $v(y) < \infty$ for all $y \in E$ then v_i are continuous and converge to v in a monotone way. Moreover, v_i is a value function for the problem with the number of transactions bounded by i.

Proof. Continuity of v_i is obtained by induction. It results from the Feller property of $\big(S(t), X(t)\big)$, continuity of the value function corresponding to a stopping problem with a bounded functional as in the above Bellman operator (for continuity and boundedness of Mv_i see Lemma 3.2, for further details see [10]). Consequently, standard impulsive control arguments show that v_i is a value function for the problem with the number of transactions bounded by i.

Although it is evident that v_i is a non-decreasing sequence, the fact that $v_i(y)$ converges to $v(y)$ for every $y \in Y$ requires justification. Fix $y \in E$ and denote by $u(y)$ the limit $\lim_{i\to\infty} v_i(y)$. Certainly, $u(y) \le v(y)$. Fix a strategy $\Pi \in \mathcal{A}(y)$ and denote by Π^i its restriction to first i transactions. We easily have that $v_i(y) \ge J(\Pi^i)$ and $u(y) \ge J(\Pi^i)$. Observe that

$$g_i = \int_0^{\tau_i} e^{-\alpha s} F\big(Y^{\Pi}(s)\big)ds + \sum_{k=1}^{i} e^{-\alpha\tau_k^{\Pi}} G(C_k^{\Pi})$$

is a monotone sequence of non-negative random variables with the limit $g = \int_0^{\infty} e^{-\alpha s} F\big(Y^{\Pi}(s)\big)ds + \sum_{k=1}^{\infty} e^{-\alpha\tau_k^{\Pi}} G(C_k^{\Pi})$. By the monotone convergence theorem $\mathbb{E}^{\,y} g_i \to \mathbb{E}^{\,y} g = J(\Pi)$. Moreover, $J(\Pi^i) \ge \mathbb{E}^{\,y} g_i$, so $\liminf_{i\to\infty} J(\Pi^i) \ge J(\Pi)$. Hence, $u(y) \ge J(\Pi)$, because $v_i(y) \ge J(\Pi^i)$. Since this is true for any $\Pi \in \mathcal{A}(y)$, $u(y) \ge v(y)$. ∎

Now we can explain the meaning of the switching operator and assumption UF. For $u = v_i$ or $u = v$ we have

$$Mu(\eta, s, x) = \begin{cases} \sup\big\{G(\varsigma) + u(\eta_1, s, x) : \quad (\eta_1, \varsigma) \in \Gamma(\eta, s)\big\}, & \Gamma(\eta, s) \neq \emptyset, \\ u(0, s, x), & \Gamma(\eta, s) = \emptyset. \end{cases}$$

Therefore, an impulse to 0 is a synonym of going bankrupt when one wants to perform a transaction without having enough money to cover transaction costs. Assumption UF assures that it cannot be economical. It is not possible to get a worse score than that related to the zero portfolio.

4. The optimal strategy for bounded F and G

We shall concentrate first on the case of bounded reward function F without consumption, i.e. $G \equiv 0$. We recall that F is assumed to be continuous and non-negative.

Assumption NFL. For every $y \in E$ there exists an $\epsilon > 0$ such that for every $T > 0$

$$H(B, T) = \sup_{z\in B} \sup_{\Pi\in\mathcal{A}(z)} \mathbb{E}^{\,z} N^{\Pi}(T) S(T) < \infty,$$

where N^{Π} is a share holding process linked to Π and $B = B\big(y, \epsilon\big) \subseteq E$ is an open ball.

Assumption NUM. S^1 is a non-decreasing process.

Assumption NFL plays the role of a no-arbitrage condition. It also implies that the market satisfies condition F. Assumption NUM states that the first asset represents a generalized bank account (positive jumps are allowed). These assumptions are required to obtain an estimate for the growth of the transactions' moments.

LEMMA 4.1. If assumptions NFL and NUM are satisfied then for any $y \in E$ there exists $\delta > 0$ such that

$$\sup_{z\in B(y,\delta)} \sup_{\Pi\in\mathcal{A}(z)} \mathbb{E}^{\,y} e^{-\alpha\tau_i^{\Pi}} \to 0, \quad \text{as } i \to \infty,$$

where $B(y, \delta)$ is an open ball in E.

Proof. Fix $T > 0$ and consider an estimate

$$\mathbb{E}^{z} e^{-\alpha\tau_i^{\Pi}} \leq \mathbb{P}^z(\tau_i^{\Pi} \leq T) + e^{-\alpha T}\mathbb{P}^z(\tau_i^{\Pi} > T) \leq \mathbb{P}^z(\tau_i^{\Pi} \leq T) + e^{-\alpha T}. \tag{6}$$

From NFL there exists a ball $B = B(2y, \epsilon)$ such that $H(B, T) < \infty$. Put $\delta = \frac{\epsilon}{2}$ and take $z = (\eta, s, x) \in B(y, \delta)$ and a strategy $\Pi \in \mathcal{A}(z)$. By Proposition 2.1 there exists a doubled strategy $\tilde{\Pi} \in \mathcal{A}(2\eta, s, x)$ (i.e. with $\alpha = 2$) with the following bound for the savings deposited in S^1:

$$\gamma_i - \gamma_{i-1} \geq \frac{\frac{1}{\beta+1}K}{S^1(\tau_i)}.$$

S^1 is non-decreasing, so $\gamma_i S^1(\tau_i) \geq i\frac{1}{\beta+1}K$. Hence,

$$\mathbb{P}^z(\tau_i^{\Pi} \leq T) = \mathbb{P}^z(\tau_i^{\tilde{\Pi}} \leq T) \leq \mathbb{P}^z\Big(N^{\tilde{\Pi}}(T)S(T) > i\frac{1}{\beta+1}K\Big) \leq \frac{\mathbb{E}^{z} N^{\tilde{\Pi}}(T)S(T)}{i\frac{1}{\beta+1}K} \leq \frac{H(B,T)}{i\frac{1}{\beta+1}K}$$

by Tchebyshev inequality. Consequently

$$\mathbb{E}^{z} e^{-\alpha\tau_i^{\Pi}} \leq \frac{H(B,T)}{i\frac{1}{\beta+1}K} + e^{-\alpha T}.$$

Therefore, $\mathbb{E}^{z} e^{-\alpha\tau_i^{\Pi}}$ tends to zero uniformly for $z \in B(y, \delta)$, $\Pi \in \mathcal{A}(z)$. ■

THEOREM 4.2. Assume NFL, NUM and $G \equiv 0$. The value function v is continuous and satisfies the Bellman equation

$$v = \mathcal{G}v.$$

Proof. First observe that v is a bounded function from boundedness of F. Let

$$v_0(y) = \mathbb{E}^{y} \int_0^{\infty} e^{-\alpha s} F\big(Y(s)\big) ds$$

and $v_i(y) = \mathcal{G}v_{i-1}(y)$, $i = 1, 2, \ldots$. By Lemma 3.3 v_i increases to v and v_i, $i = 0, 1, \ldots$, are continuous functions. We shall show that the convergence is almost uniform (i.e. uniform on compacts), which implies that the value function is continuous. Fix $y \in E$. Let us write Π^i for the restriction of the strategy Π to the first i impulses. Put $\mathcal{A}^i(y) = \{\Pi^i : \quad \Pi \in \mathcal{A}(y)\}$. We get the following estimate:

$$\begin{aligned} 0 \leq v(y) - v_i(y) &= \sup_{\Pi \in \mathcal{A}(y)} J(\Pi) - \sup_{\tilde{\Pi} \in \mathcal{A}_i(y)} J(\tilde{\Pi}) \leq \sup_{\Pi \in \mathcal{A}(y)} \big|J(\Pi) - J(\Pi^i)\big| \\ &\leq \sup_{\Pi \in \mathcal{A}(y)} \frac{\|F\|_\infty}{\alpha} \mathbb{E}^{y} e^{-\alpha\tau_i^{\Pi}} \end{aligned} \tag{7}$$

By Lemma 4.1 $\mathbb{E}^{y} e^{-\alpha\tau_i^{\Pi}}$ tends to zero uniformly on a ball $B(y, \delta)$ for some $\delta > 0$. Therefore, the convergence of v_i to v on this ball is uniform. Consequently, v is a continuous function.

The proof that $v = \mathcal{G}v$ is straightforward. ■

The first part of this section is devoted to the case with $G \equiv 0$, i.e. without consumption. Here we move to the general case. This, however, requires an additional assumption NFL1, introduced below. Although consumption can be easily incorporated into the above theorem, we shall present a different proof of existence of an optimal strategy based on a classical result concerning solution of the Bellman equation. Let us introduce the following notation:

$$H_\epsilon(B,T) = \sup_{z\in B}\sup_{\Pi\in\mathcal{A}(z)} \mathbb{E}^{\,z}\big(N^{\Pi}(T)\cdot S\big(T\big)\big)^{1+\epsilon}, \quad \epsilon>0$$

for a compact set $B\subseteq E$ and $T>0$.

Assumption NFL1. For every $y\in E$ there exists constants $\delta,\theta,\kappa,\epsilon$ with $\kappa<\alpha$ such that

$$H_\epsilon(B,t)\le \theta e^{\kappa t},$$

where $B=B(y,\delta)$.

LEMMA 4.3. If NUM and NFL1 are satisfied then for every $y\in E$ there exists $\delta>0$ such that

$$\sum_{i=1}^{\infty}\sup_{z\in B(y,\delta)}\sup_{\Pi\in\mathcal{A}(z)}\mathbb{E}^{\,\Pi}e^{-\alpha\tau_i^{\Pi}}<\infty$$

and consequently

$$\sup_{z\in B(y,\delta)}\sup_{\Pi\in\mathcal{A}(z)}\sum_{i=1}^{\infty}\mathbb{E}^{\,\Pi}e^{-\alpha\tau_i^{\Pi}}<\infty.$$

Proof. Fix $y\in E$. By assumption NFL1 there exist $B=B(2y,2\delta)$ and positive constants θ,κ,ϵ, with $\kappa<\alpha$, such that

$$H_\epsilon(B,t)\le\theta e^{\kappa t},\quad t\ge 0. \tag{8}$$

Take $z=(\eta,s,x)\in B(y,\delta)$ and a strategy $\Pi\in\mathcal{A}(z)$. By Proposition 2.1 there exists a doubled strategy $\tilde{\Pi}\in\mathcal{A}(2\eta,s,x)$ (i.e. with $\alpha=2$) with the following bound for savings invested in S^1

$$\gamma_i-\gamma_{i-1}\ge\frac{\frac{1}{\beta+1}K}{S^1(\tau_i)}.$$

S^1 is non-decreasing, so $\gamma_i S^1(\tau_i)\ge i\frac{1}{\beta+1}K$. By Tchebyshev inequality, for any $t\ge0$,

$$\begin{aligned}\mathbb{P}^y(\tau_i^{\Pi}\le t)=\mathbb{P}^y(\tau_i^{\tilde{\Pi}}\le t)&\le\mathbb{P}^y\Big(N^{\tilde{\Pi}}(t)\cdot S(t)>i\frac{1}{\beta+1}K\Big)\\&\le\frac{\mathbb{E}^{\,y}\big(N^{\tilde{\Pi}}(t)\cdot S(t)\big)^{1+\epsilon}}{\big(i\frac{1}{\beta+1}K\big)^{1+\epsilon}}\le\frac{H_\epsilon(B,t)}{\big(i\frac{1}{\beta+1}K\big)^{1+\epsilon}},\end{aligned}$$

and the bound does not depend on the choice of $\Pi\in\mathcal{A}(z)$, $z\in B(y,\delta)$. Integration by parts yields the following estimate

$$\mathbb{E}^{\,y}e^{-\alpha\tau_i^{\Pi}}=\alpha\int_0^\infty e^{-\alpha t}\,\mathbb{P}(\tau_i^{\Pi}\le t)dt\le\alpha\int_0^\infty e^{-\alpha t}\frac{H_\epsilon(B,t)}{\big(i\frac{1}{\beta+1}K\big)^{1+\epsilon}}\,dt=\frac{D\alpha}{i^{1+\epsilon}}\frac{1}{\alpha-\kappa}$$

with

$$D=\frac{\theta}{\big(\frac{1}{\beta+1}K\big)^{1+\epsilon}}.$$

Hence

$$\sum_{i=1}^{\infty}\sup_{z\in B(y,\delta)}\sup_{\Pi\in\mathcal{A}(z)}\mathbb{E}^{\,y}e^{-\alpha\tau_i^{\Pi}}\le\frac{D\alpha}{\alpha-\kappa}\sum_{i=1}^{\infty}\frac{1}{i^{1+\epsilon}}<\infty. \tag{9}$$

■

THEOREM 4.4. Assume that NUM and NFL1 are satisfied. The value function $v(y)$ is continuous and satisfies the Bellman equation $v = \mathcal{G}v$.

Proof. Let

$$v_0(y) = \mathbb{E}^{\,y} \int_0^\infty e^{-\alpha s} F\big(Y(s)\big) ds$$

and $v_i(y) = \mathcal{G}v_{i-1}(y)$, $i = 1, 2, \ldots$. By Lemma 3.3 v_i increases to v and v_i, $i = 0, 1, \ldots$, are continuous functions. We shall show that the convergence is uniform on compacts, which implies that the value function is continuous. Fix $y \in E$. Let us write Π^i for the restriction of the strategy Π to the first i impulses. Put $\mathcal{A}^i(y) = \{\Pi^i : \quad \Pi \in \mathcal{A}(y)\}$. We obtain the following estimate:

$$\begin{aligned} 0 \le v(y) - v_i(y) &= \sup_{\Pi \in \mathcal{A}(y)} J(\Pi) - \sup_{\tilde{\Pi} \in \mathcal{A}_i(y)} J(\tilde{\Pi}) \le \sup_{\Pi \in \mathcal{A}(y)} \big| J(\Pi) - J(\Pi^i) \big| \\ &\le \sup_{\Pi \in \mathcal{A}(y)} \Big\{ \frac{\|F\|_\infty}{\alpha} \mathbb{E}^{\,y} e^{-\alpha \tau_i^\Pi} + \|G\|_\infty \sum_{k=i}^\infty \mathbb{E}^{\,y} e^{-\alpha \tau_k^\Pi} \Big\}. \end{aligned} \tag{10}$$

Hence $v_i(y)$ converges to $v(y)$ uniformly on compact subsets of E by Lemma 4.1 and 4.3. Consequently, v is a continuous function. By the monotonicity of the sequence v_i and Monotone Convergence Theorem we obtain $v = \mathcal{G}v$. ■

COROLLARY 4.5. Under assumptions of the above theorems there exists an optimal Markovian strategy maximizing the reward functional. It is characterised by an impulse region $\mathcal{I}$ given by

$$\mathcal{I} = \{y \in E : \quad v(y) = Mv(y)\},$$

where v is a value function. In detail, for a starting point $y = (\eta, s, x)$

$$\begin{aligned} \tau_1 &= \inf\{t \ge 0 : \quad \big(\eta, S(t), X(t)\big) \in \mathcal{I}\}, \\ \tau_i &= \inf\{t > \tau_{i-1} : \quad \big(N_{i-1}, S(t), X(t)\big) \in \mathcal{I}\}, \\ N_i &= I_1\big(N_{i-1}, S(\tau_i), X(\tau_i)\big), \\ C_i &= I_2\big(N_{i-1}, S(\tau_i), X(\tau_i)\big), \end{aligned}$$

where $I : E \to E \times [0, \infty)$ is the impulse function for Mv.

Proof. The proof is an easy consequence of Lemma 3.2 and 3.3. ■

Corollary 4.5 states that we are able to find an optimal strategy. Moreover, this strategy is Markovian, i.e. it depends only on the present state of the world. It is also stationary: there are equivalent requirements for each transaction to take place. Observe that for $G \equiv 0$ (Theorem 4.2) the second coordinate of the impulse function I_2 is equal to zero and no wealth is consumed in the optimal strategy.

5. Unbounded reward functional

In the previous section we assumed F and G to be bounded. Now we remove this restriction; functions F and G are unbounded, non-negative and continuous. However, we will need a new assumption which takes into account functions F, G. Recall that neither NFL nor NFL1 depend on F, G.

Assumption NFL2. For every $y \in E$ there exist $\kappa > 1$, $\epsilon > 0$ such that

$$\sup_{z\in B} \sup_{\Pi\in\mathcal{A}(z)} \mathbb{E}^{\,z}\Big\{ \int_0^\infty e^{-\alpha t}\Big(F\big(Y^\Pi(t)\big)\Big)^\kappa dt + \sum_{k=1}^\infty e^{-\alpha\tau_k^\Pi}\big(G(C_k^\Pi)\big)^\kappa\Big\} < \infty,$$

where $B = B\big(y,\epsilon\big) \subseteq E$ is an open ball.

THEOREM 5.1. If NUM, NFL1 and NFL2 are satisfied then the value function v is continuous and satisfies Bellman equation $v = \mathcal{G}v$. Moreover, there exists an optimal strategy.

REMARK 5.2. If $G \equiv 0$, i.e. there is no consumption, assumption NFL1 can be replaced with NFL. Observe also that NFL1 implies NFL.

Proof of theorem 5.1. Assumptions NFL1 and NFL2 guarantee that $v(y) < \infty$ for all $y \in E$. Let

$$v_0(y) = \mathbb{E}^{\,y} \int_0^\infty e^{-\alpha s} F\big(Y(s)\big)ds$$

and $v_i(y) = \mathcal{G}v_{i-1}(y)$, $i = 1, 2, \ldots$. In the contrary to the proof of Theorem 4.4 we cannot use Lemma 3.3 to prove continuity of v_i because of unboundedness of F and G. Instead, for $\Pi \in \mathcal{A}(y)$, $y \in E$, we define a family of functionals

$$J^H(\Pi) = \mathbb{E}^{\,(s,x)}\Big\{ \int_0^\infty e^{-\alpha t}\big(F\big(Y^\Pi(t)\big) \wedge H\big)dt + \sum_{k=1}^\infty e^{-\alpha\tau_k^\Pi}\big(G\big(C_k^\Pi\big) \wedge H\big)\Big\}, \quad H > 0.$$

Denote by v^H and v_i^H, $i = 0, 1, \ldots$, appropriate value functions. By Theorem 4.4 they are continuous. We shall show that they converge uniformly on compact sets to v and v_i respectively as H tends to ∞.

Fix $y \in E$. By assumption NFL2 there exist constants $\epsilon > 0$ and $\kappa > 1$ such that

$$\begin{aligned} &\sup_{z\in B(y,\epsilon)} \sup_{\Pi\in\mathcal{A}(z)} \mathbb{E}^{\,z} \int_0^\infty e^{-\alpha t}\Big(F\big(Y^\Pi(t)\big)\Big)^\kappa dt < D < \infty, \\ &\sup_{z\in B(y,\epsilon)} \sup_{\Pi\in\mathcal{A}(z)} \mathbb{E}^{\,z} \sum_{k=1}^\infty e^{-\alpha\tau_k^\Pi}\Big(G(C_k^\Pi)\Big)^\kappa < D < \infty. \end{aligned} \tag{11}$$

For $z \in B(y,\epsilon)$, $H > 0$ define

$$R^H(z) = \sup_{\Pi\in\mathcal{A}(z)} \mathbb{E}^{\,z}\Big\{ \int_0^\infty e^{-\alpha t} F\big(Y^\Pi(t)\big)1_{F(Y^\Pi(t))\geq H}dt + \sum_{k=1}^\infty e^{-\alpha\tau_k^\Pi} G\big(C_k^\Pi\big)1_{G(C_k^\Pi)\geq H}\Big\} \tag{12}$$

and observe that

$$\begin{aligned} v_i(z) - v_i^H(z) &\leq R^H(z), \\ v(z) - v^H(z) &\leq R^H(z). \end{aligned}$$

For the first term of (12) we obtain

$$\begin{aligned} &\mathbb{E}^{\,z} \int_0^\infty e^{-\alpha t} F\big(Y^\Pi(t)\big)1_{F(Y^\Pi(t))\geq H}dt \\ &\qquad\leq \mathbb{E}^{\,z} \int_0^\infty e^{-\alpha t} F\big(Y^\Pi(t)\big)\frac{\big(F\big(Y^\Pi(t)\big)\big)^{\kappa-1}}{H^{\kappa-1}}1_{F(Y^\Pi(t))\geq H}dt \\ &\qquad\leq \frac{1}{H^{\kappa-1}}\mathbb{E}^{\,z} \int_0^\infty e^{-\alpha t}\big(F\big(Y^\Pi(t)\big)\big)^\kappa dt. \end{aligned}$$

Similar argument applied to the second term of (12) yields

$$\mathbb{E}^{z}\sum_{k=1}^{\infty}e^{-\alpha\tau_k^{\Pi}}G\big(C_k^{\Pi}\big)1_{G(C_k^{\Pi})\geq H}\leq\frac{1}{H^{\kappa-1}}\mathbb{E}^{z}\sum_{k=1}^{\infty}e^{-\alpha\tau_k^{\Pi}}\big(G\big(C_k^{\Pi}\big)\big)^{\kappa}.$$

Hence, by (11)

$$R^H(z)\leq\frac{2D}{H^{\kappa-1}},\quad z\in B(y,\epsilon).$$

Consequently, v_i and v are continuous and v_i converges to v pointwise as $i\to\infty$. Moreover, v_i is a value function for the problem with the number of transactions bounded by i. We show that $v=\mathcal{G}v$ in the same way as in the proof of Theorem 4.4. The optimality of the strategy Π constructed in Corollary 4.5 results from pointwise convergence of v_i to v. ■

6. Examples

In Subsection 6a we provide sufficient conditions for NFL, NFL1 and NFL2 to be satisfied in a general Markovian multiplicative model. Subsection 6b elaborates on a specific case of the multiplicative model, namely exponential diffusion with bounded coefficients depending on Markovian economic factors. We are able to provide stronger results than those coming from Subsection 6a. Important examples of reward functions satisfying assumptions UF and AF are given in Subsection 6c.

6a. Multiplicative price process

The market is modeled by the d-dimensional price process $S(t)$ given as

$$S^i(t)=e^{Z^i(t)},\quad i=1,\dots,d.$$

Let $X(t)$ be the process of economic factors. Assume that $\big(Z(t),X(t)\big)$ and $\big(S(t),X(t)\big)$ satisfy the Feller property.

The aim of this subsection is to formulate sufficient conditions in terms of the underlying price process for NFL, NFL1 and NFL2 to hold true.

Assumption MLT. There exist $\beta>0$

$$\mathbb{E}^{\tilde{z}}\big\{\big(S^i(t)\big)^2\big|\mathcal{F}_u\big\}\leq\big(S^i(u)\big)^2e^{2\beta(t-u)},\quad 0\leq u\leq t,\quad i=1,\dots,d,\quad \tilde{z}\in E.$$

The diffusion price process given by equation (21) in the next subsection satisfies the MLT with β equal to that of Lemma 6.14.

THEOREM 6.1. If the price process satisfies MLT, there exist $\beta>0$ such that

$$\sup_{\Pi\in\mathcal{A}(y)}\mathbb{E}^{y}\big(N^{\Pi}(T)\cdot S(T)\big)^2\leq(\eta_0\cdot s_0)^2e^{2\beta T},\quad y=(\eta_0,s_0,x_0)\in E,\quad T\geq 0.$$

COROLLARY 6.2. If the price process satisfies MLT, there exist $\beta>0$ such that

$$\sup_{\Pi\in\mathcal{A}(y)}\mathbb{E}^{y}\big(N^{\Pi}(T)\cdot S(T)\big)^{\delta}\leq(\eta_0\cdot s_0)^{\delta}e^{\delta\beta T},\quad y=(\eta_0,s_0,x_0)\in E,\quad T\geq 0,\quad \delta\in[1,2].$$

Proof. It is an easy application of Hölder inequality. ■

We shall postpone the proof of Theorem 6.1 in order to concentrate first on its consequences. Theorem 6.1 and Corollary 6.2 enable us to find sufficient conditions for assumptions NFL, NFL1 and NFL2 to be satisfied. Assume MLT with a constant β. Easily, for $T > 0$, $z = (\eta_0, s_0, x_0) \in E$, $\Pi \in \mathcal{A}(z)$

$$\mathbb{E}^{\,z} N^{\Pi}(T) \cdot S(T) \leq (\eta_0 \cdot s_0) e^{\beta T} \tag{13}$$

so assumption NFL is satisfied. Assumption NFL1 is satisfied for any $\epsilon \in (0, 1]$ with $\kappa = (1 + \epsilon)\beta$, since by Corollary 6.2 for $z = (\eta_0, s_0, x_0) \in E$

$$\sup_{\Pi \in \mathcal{A}(z)} \mathbb{E}^{\,z} \big(N^{\Pi}(T) \cdot S(T)\big)^{1+\epsilon} \leq (\eta_0 \cdot s_0)^{1+\epsilon} e^{(1+\epsilon)\beta t}. \tag{14}$$

Only the case of unbounded reward functional requires more consideration. Assume that $\alpha > \beta$, G is bounded and F satisfies

Assumption AF. There exist constants A, B such that

$$0 \leq F(\eta, s, x) \leq A + B(\eta \cdot s), \quad (\eta, s, x) \in E.$$

We shall show that NFL2 is satisfied with $\kappa \in (1, \frac{\alpha}{\beta} \wedge 2]$.

LEMMA 6.3. There exist constants $C_1, C_2 \geq 0$ depending on $\kappa \in [1, \frac{\alpha}{\beta} \wedge 2]$ such that for $y = (\eta_0, s_0, x_0) \in E$,

$$\sup_{\Pi \in \mathcal{A}(y)} \mathbb{E}^{\,z} \int_0^\infty e^{-\alpha t} F\big(Y^{\Pi}(t)\big)^{\kappa} dt \leq C_1 + C_2 (\eta_0 \cdot s_0)^{\kappa}.$$

Proof. By Corollary 6.2 and the fact that $a \leq 1 + a^2$ for $a \geq 0$, for $\gamma > 2\beta$

$$\begin{aligned}
&\sup_{\Pi \in \mathcal{A}(y)} \mathbb{E}^{\,y} \int_0^\infty e^{-\gamma t} F\big(Y^{\Pi}(t)\big)^2 dt \\
&\leq \sup_{\Pi \in \mathcal{A}(y)} \mathbb{E}^{\,y} \int_0^\infty e^{-\gamma t} \Big(A^2 + 2AB N^{\Pi}(t) \cdot S(t) + B^2 \big(N^{\Pi}(t) \cdot S(t)\big)^2\Big) dt \\
&= \frac{A^2}{\gamma} + 2AB\, \mathbb{E}^{\,y} \int_0^\infty e^{-\gamma t} N^{\Pi}(t) \cdot S(t) dt + B^2\, \mathbb{E}^{\,y} \int_0^\infty e^{-\gamma t} \big(N^{\Pi}(t) \cdot S(t)\big)^2 dt \\
&\leq \frac{A^2}{\gamma} + 2AB \Big(\frac{1}{\gamma} + (\eta_0 \cdot s_0)^2 \int_0^\infty e^{-\gamma t + 2\beta t} dt\Big) + B^2 (\eta_0 \cdot s_0)^2 \int_0^\infty e^{-\gamma t + 2\beta t} dt \\
&= \tilde{C}_1 + \tilde{C}_2 (\eta_0 \cdot s_0)^2.
\end{aligned}$$

Take $\gamma = \frac{2\alpha}{\kappa}$. By Hölder inequality we obtain

$$\sup_{\Pi \in \mathcal{A}(y)} \mathbb{E}^{\,y} \int_0^\infty e^{-\alpha t} F\big(Y^{\Pi}(t)\big)^{\kappa} dt \leq \big(\tilde{C}_1 + \tilde{C}_2 (\eta_0 \cdot s_0)^2\big)^{\kappa/2}.$$

We conclude by the application of the inequality $(a + b)^{\delta} \leq a^{\delta} + b^{\delta}$ for $\delta \in (0, 1]$, $a, b \geq 0$. ■

We shall prove NFL2 for any $\kappa \in (1, \frac{\alpha}{\beta} \wedge 2]$. By Lemma 6.3 the first term of NFL2

$$\sup_{\Pi \in \mathcal{A}(y)} \mathbb{E}^{\,y} \int_0^\infty e^{-\alpha t} F\big(Y^{\Pi}(t)\big)^{\kappa} dt$$

is bounded uniformly for y in compact subsets of E. The second term is also uniformly bounded on compact sets by Lemma 4.3, since for $y \in E$

$$\sup_{\Pi \in \mathcal{A}(y)} \mathbb{E}^{\,z} \sum_{k=1}^{\infty} e^{-\alpha \tau_k^\Pi} \big(G(C_k^\Pi)\big)^\kappa \le \big(\sup_{\varsigma \in [0,\infty)} G(\varsigma)\big)^\kappa \sup_{\Pi \in \mathcal{A}(y)} \mathbb{E}^{\,z} \sum_{k=1}^{\infty} e^{-\alpha \tau_k^\Pi}.$$

Now, let us concentrate on the proof of Theorem 6.1. Notice that it is sufficient to prove the estimate on the wealth in the case of no transaction costs but with the same form of strategies. A class of strategies we deal with here is described in the following definition. It differs from the class we worked with only in two points. We removed consumption and the requirement of financing transaction costs. We may think of transaction costs being funded from external sources.

DEFINITION 6.4. $\Pi = \big((N_0, \tau_0), (N_1, \tau_1), \dots\big)$ is an **admissible strategy without transaction costs** if

i) $0 = \tau_0 \le \tau_1 \le \tau_2 \cdots$ are stopping times,
ii) N_i is $\mathcal{F}_{\tau_i}$ measurable,
iii) $N_i \in E^N$ a.s.,
iv) $N_i S(\tau_i) = N_{i-1} S(\tau_i)$ a.s. (self-financing),
v) $\mathbb{P}(\lim_{n\to\infty} \tau_n = \infty) = 1$.

The set of admissible strategies for a starting point $y = (\eta_0, s_0, x_0) \in E$ is denoted by $\tilde{\mathcal{A}}(y)$. An admissible strategy Π belongs to $\tilde{\mathcal{A}}(y)$ if $N_0 = \eta_0$ and the self-financing condition holds for the price process starting from (s_0, x_0). Now we can formulate the result.

PROPOSITION 6.5. If the price process satisfies MLT with the constant β then for each $T \ge 0$

$$\sup_{\Pi \in \tilde{\mathcal{A}}(y)} \mathbb{E}^{\,y} \big(N^\Pi(T) \cdot S(T)\big)^2 \le (\eta_0 \cdot s_0)^2 e^{2\beta T}, \quad y = (\eta_0, s_0, x_0) \in E. \tag{15}$$

The proof of Proposition 6.5 is based on the time discretization and continuity of the semigroup corresponding to the Feller processes. First we prove (15) for strategies with transactions occurring only in a finite number of deterministic times. As the set of this moments gets more dense in $[0, T]$ its approximation of the left-hand side of (15) improves. Finally, in the limit we obtain (15).

Proof of Proposition 6.5. We will skip the superscript Π in N^Π and τ_i^Π if it does not cause ambiguity. For $n \in \mathbb{N}$ we define a dyadic split of the interval $[0, T]$

$$D_n = \Big\{0, \frac{T}{2^n}, \frac{2T}{2^n}, \dots, T\Big\}.$$

In the set of admissible strategies without transaction costs $\tilde{\mathcal{A}}(y)$ we choose those with transactions held almost surely in the set D_n and denote them by $\tilde{\mathcal{A}}_n(y)$. We can view a strategy in $\tilde{\mathcal{A}}_n(y)$ as having transactions at each point of D_n. It is justified by the absence of transaction costs, since any change in the portfolio contents can take place only in the points of D_n. By $\tilde{\mathcal{A}}^m(y)$ we understand all trading strategies that have at most m transactions on $[0, T]$. Similarly, $\tilde{\mathcal{A}}_n^m(y)$ denotes the strategies from $\tilde{\mathcal{A}}^m(y)$ that belong to $\tilde{\mathcal{A}}_n(y)$. More formally

$$\begin{aligned}
\tilde{\mathcal{A}}_n(y) &= \big\{\Pi \in \tilde{\mathcal{A}}(y) : \quad \mathbb{P}^y(\tau_i \cap [0, T] \in D_n \ \forall i) = 1\big\}, \quad y \in E, \quad n \in \mathbb{N},\\
\tilde{\mathcal{A}}^m(y) &= \big\{\Pi \in \tilde{\mathcal{A}}(y) : \quad \tau_{m+1} > T \quad \mathbb{P}^y\text{-a.s.}\big\}, \quad y \in E, \quad m \in \mathbb{N},\\
\tilde{\mathcal{A}}_n^m(y) &= \tilde{\mathcal{A}}^m(y) \cap \tilde{\mathcal{A}}_n(y), \quad y \in E, \quad m \in \mathbb{N}.
\end{aligned}$$

The following lemma assures integrability of the left-hand side of (15) for strategies from $\tilde{\mathcal{A}}^m$.

LEMMA 6.6.
$$\mathbb{E}^{\,y}\big(N^{\Pi}(T)\cdot S(T)\big)^2<\infty,\quad \Pi\in\tilde{\mathcal{A}}^m(y),\quad m\in\mathbb{N},\quad y\in E.$$

Proof. We shall obtain the estimate
$$\mathbb{E}^{\,y}\big(N^{\Pi}(T)\cdot S(T)\big)^2\le(\eta_0\cdot s_0)^2e^{2m\beta T},\tag{16}$$
where $y=(\eta_0,s_0,x_0)$ and β is the constant from assumption MLT.

First notice that $\big(N^{\Pi}(T)\cdot S(T)\big)^2$ is a non-negative random variable, so its conditional expectation is well-defined, though it can be infinite in some points. Fix $m\in\mathbb{N}$. The lack of transaction costs allows us to assume that $\tau_m\le T$. It may be done formally in the following way:
$$\begin{aligned}&\tau_1'=\tau_1\wedge T,\dots,\tau_m'=\tau_m\wedge T,\\ &\tau_{m+1}'=\tau_1\vee T,\dots\tau_{2m}'=\tau_m\vee T,\\ &\tau_{2m+k}'=\tau_{m+k},\quad k>0.\end{aligned}$$
Obviously, the portfolio contents process linked to the new trading strategy is identical to $N^{\Pi}(t)$.

Therefore, with the above transformation in mind we obtain
$$\begin{aligned}\mathbb{E}^{\,y}\big(N^{\Pi}(T)\cdot S(T)\big)^2&=\mathbb{E}^{\,y}\Big\{\sum_{i=1}^{d}\sum_{j=1}^{d}N^{\Pi}(T)^iS^i(T)N^j(T)S^j(T)\Big\}\\ &=\mathbb{E}^{\,y}\Big\{\sum_{i=1}^{d}\sum_{j=1}^{d}\mathbb{E}^{\,y}\big\{N^i(T)S^i(T)N^j(T)S^j(T)\big|\mathcal{F}_{\tau_m}\big\}\Big\}.\end{aligned}$$
We estimate separately each element of the sum using Schwarz inequality.
$$\begin{aligned}\mathbb{E}^{\,y}\big\{N^i(T)S^i(T)N^j(T)S^j(T)\big|\mathcal{F}_{\tau_m}\big\}&=\mathbb{E}^{\,y}\big\{N^i(\tau_m)S^i(T)N^j(\tau_m)S^j(T)\big|\mathcal{F}_{\tau_m}\big\}\\ &=N^i(\tau_m)N^j(\tau_m)\mathbb{E}^{\,y}\big\{S^i(T)S^j(T)\big|\mathcal{F}_{\tau_m}\big\}\\ &\le N^i(\tau_m)N^j(\tau_m)\Big(\mathbb{E}^{\,y}\big\{\big(S^i(T)\big)^2\big|\mathcal{F}_{\tau_m}\big\}\Big)^{\frac12}\\ &\qquad\Big(\mathbb{E}^{\,y}\big\{\big(S^j(T)\big)^2\big|\mathcal{F}_{\tau_m}\big\}\Big)^{\frac12}.\end{aligned}$$
Assumption MLT allows us to make the following bound
$$\begin{aligned}N^i(\tau_m)N^j(\tau_m)S^i(\tau_m)\Big(\mathbb{E}^{\,y}\big\{e^{2\beta(T-\tau_m)}\big|\mathcal{F}_{\tau_m}\big\}\Big)^{\frac12}S^j(\tau_m)\Big(\mathbb{E}^{\,y}\big\{e^{2\beta(T-\tau_m)}\big|\mathcal{F}_{\tau_m}\big\}\Big)^{\frac12}\\ \le N^i(\tau_m)N^j(\tau_m)S^i(\tau_m)S^j(\tau_m)e^{2\beta T}.\end{aligned}$$
Finally
$$\begin{aligned}\mathbb{E}^{\,y}\big(N^{\Pi}(T)\cdot S(T)\big)^2&\le\mathbb{E}^{\,y}\Big\{\sum_{i=1}^{d}\sum_{j=1}^{d}N^i(\tau_m)N^j(\tau_m)S^i(\tau_m)S^j(\tau_m)e^{2\beta T}\Big\}\\ &=\mathbb{E}^{\,y}\big(N(\tau_m)\cdot S(\tau_m)\big)^2e^{2\beta T}.\end{aligned}$$
The condition of self-financing of the strategy yields
$$\mathbb{E}^{\,y}\big(N(\tau_m)\cdot S(\tau_m)\big)^2e^{2\beta T}\le\mathbb{E}^{\,y}\big(N(\tau_m-)\cdot S(\tau_m)\big)^2e^{2\beta T}.$$

Repeating this reasoning leads to (16). ■

In the following lemma we use the fact that the strategies from $\tilde{\mathcal{A}}_n(y)$ can be perceived as having transactions in each moment of the dyadic split D_n. Let us denote by β the constant from the estimate in assumption MLT.

LEMMA 6.7. For each $n \in \mathbb{N}$

$$\mathbb{E}^{\,y}\big(N^{\Pi}(T) \cdot S(T)\big)^2 \leq (\eta_0 \cdot s_0)^2 e^{2\beta T}, \quad \Pi \in \tilde{\mathcal{A}}_n(y), \quad y = (\eta_0, s_0, x_0) \in E.$$

Proof. Fix $y \in E$ and a strategy $\Pi \in \mathcal{A}_n(y)$. For use in this proof we denote all points of D_n by $t_0, t_1, \ldots, t_{2^n}$, i.e. $t_i = Ti/2^n$. Furthermore, we use the notation $N_{2^n} = N^{\Pi}(t_{2^n})$, $N_{2^n-1} = N^{\Pi}(t_{2^n-1}), \ldots$. Clearly, N_{t_i} is $\mathcal{F}_{t_i}$ measurable for $i = 1, \ldots 2^n$.

By self-financing condition we have that $N_{2^n} \cdot S(T) = N_{2^n-1} \cdot S(T)$ and therefore

$$\begin{aligned}\mathbb{E}^{\,y}\big(N^{\Pi}(T) \cdot S(T)\big)^2 &= \mathbb{E}^{\,y}\big(N_{2^n} \cdot S(T)\big)^2 = \mathbb{E}^{\,y}\big(N_{2^n-1} \cdot S(T)\big)^2 \\ &= \mathbb{E}^{\,y}\Big\{\sum_{i=1}^{d}\sum_{j=1}^{d} N^i_{2^n-1} S^i(T) N^j_{2^n-1} S^j(T)\Big\} \\ &= \mathbb{E}^{\,y}\Big\{\sum_{i=1}^{d}\sum_{j=1}^{d} \mathbb{E}^{\,y}\big\{N^i_{2^n-1} S^i(T) N^j_{2^n-1} S^j(T)\big|\mathcal{F}_{t_{2^n-1}}\big\}\Big\}.\end{aligned} \tag{17}$$

We estimate separately each element of the sum using Schwarz inequality.

$$\begin{aligned}\mathbb{E}^{\,y}\big\{N^i_{2^n-1} S^i(T) N^j_{2^n-1} S^j(T)\big|\mathcal{F}_{t_{2^n-1}}\big\} &= N^i_{2^n-1} N^j_{2^n-1} \mathbb{E}^{\,y}\big\{S^i(T) S^j(T)\big|\mathcal{F}_{t_{2^n-1}}\big\} \\ &\leq N^i_{2^n-1} N^j_{2^n-1}\Big(\mathbb{E}^{\,y}\big\{\big(S^i(T)\big)^2\big|\mathcal{F}_{t_{2^n-1}}\big\}\Big)^{\frac{1}{2}} \\ &\qquad \Big(\mathbb{E}^{\,y}\big\{\big(S^j(T)\big)^2\big|\mathcal{F}_{t_{2^n-1}}\big\}\Big)^{\frac{1}{2}}.\end{aligned} \tag{18}$$

Assumption MLT allows us to make the following bound for (18)

$$N^i_{2^n-1} N^j_{2^n-1} S^i(t_{2^n-1}) e^{\frac{\beta}{2^n}} S^j(t_{2^n-1}) e^{\frac{\beta}{2^n}}.$$

Finally (17) is bounded by

$$\mathbb{E}^{\,y}\Big\{\sum_{i=1}^{d}\sum_{j=1}^{d} N^i_{2^n-1} S^i(t_{2^n-1}) N^j_{2^n-1} S^j(t_{2^n-1}) e^{\frac{2\beta}{2^n}}\Big\} = \mathbb{E}^{\,y}\big(N_{2^n-1} \cdot S(t_{2^n-1})\big)^2 e^{\frac{2\beta}{2^n}}.$$

Repeating the reasoning presented above $(2^n - 1)$ times we obtain the result of the lemma. ■

We shall strongly exploit the continuity of the semigroup connected to the process $\big(Z(t), X(t)\big)$. For the sake of simplicity of formulation of the following two lemmas let us denote $\tilde{Y}(t) = \big(Z(t), X(t)\big) \in \mathbb{R}^d \times \mathbb{R}^m$. The statements of the lemmas, together with the references for proof, come from [13].

LEMMA 6.8. For any compact set $K \subseteq \mathbb{R}^d \times \mathbb{R}^m$, $\epsilon > 0$, $T > 0$, there exists a compact set $K_1 \supseteq K$ such that

$$\sup_{y \in [0,\infty)^d \times K} \mathbb{P}^y\big\{\tilde{Y}(t) \notin K_1 \quad \text{for some } t \in [0, T]\big\} < \epsilon.$$

LEMMA 6.9. Let $O_\delta(\tilde{y}) = \{\tilde{z} \in \mathbb{R}^d \times \mathbb{R}^m : \quad \|\tilde{y} - \tilde{z}\| < \delta\}$, $y \in \mathbb{R}^d \times \mathbb{R}^m$.

$$\forall_{\epsilon>0,\ \delta>0} \forall_{K_1\text{ – compact}} \exists_{h_0} \forall_{h \le h_0} \forall_{\tilde{y} \in K_1} \quad \mathbb{P}^{(\eta_0,\tilde{y})}\big\{\tilde{Y}(h) \notin O_\delta(\tilde{y})\big\} < \epsilon, \quad \eta_0 \in [0,\infty)^d.$$

Lemma 6.8 is proved in [9], Lemma 2. Lemma 6.9 can be proved almost identically to Lemma 2.5 in [5]. As a corollary to Lemma 6.9 one can obtain

COROLLARY 6.10. Let τ be a bounded stopping time. For every $\delta, \epsilon > 0$, a compact set $\tilde{B} \subseteq \mathbb{R}^d \times \mathbb{R}^m$ there exists $\eta > 0$ and a compact set $\tilde{B}_1 \supseteq \tilde{B}$ such that

$$\mathbb{P}^y\big\{\|\tilde{Y}(\tau) - \tilde{Y}(\sigma)\| \ge \delta, \quad \tilde{Y}(\tau) \in \tilde{B}_1\big\} \le \epsilon, \quad y \in [0,\infty)^d \times B.$$

for any $\mathcal{F}_\tau$-measurable random variable σ satisfying

$$0 \le \sigma - \tau \le \eta.$$

Now, we will use the continuity results from Corollary 6.10 and Lemma 6.7 to obtain a bound for the wealth of portfolios from $\tilde{\mathcal{A}}^m(y)$.

LEMMA 6.11. For $m \in \mathbb{N}$, $y \in E$, $\Pi \in \tilde{\mathcal{A}}^m(y)$

$$\mathbb{E}^y\big(N^\Pi(T) \cdot S(T)\big)^2 \le \lim_{n\to\infty} \sup_{\Pi' \in \tilde{\mathcal{A}}^m_n(y)} \mathbb{E}^y\big(N^{\Pi'}(T) \cdot S(T)\big)^2.$$

Proof. We shall use the following discretization scheme of a trading strategy. Let $\Pi \in \tilde{\mathcal{A}}^m(y)$,

$$\Pi = \big((N_0,\tau_0),(N_1,\tau_1),\dots,(N_m,\tau_m)\big).$$

As before, we assume that $\tau_m \le T$. We construct an n-discretization of Π, denoted by Π_n, in order to obtain an element of $\tilde{\mathcal{A}}^m_n(y)$. Consider the following approximation of the transaction moments:

$$\tau_{n,l} = \frac{kT}{2^n} \qquad \text{if } \frac{(k-1)T}{2^n} < \tau_l \le \frac{kT}{2^n}, \qquad k = 0,1,\dots,2^n, \quad l = 0,1,\dots,m.$$

Clearly, $0 \le \tau_{n,l} - \tau_n \le 2^{-n}$. We assume that the number of assets held is proportional to that of the strategy Π, i.e.

$$N_{n,l} = \alpha_{n,l} N_l, \quad l = 0,1,\dots,m.$$

Certainly, $\alpha_{n,0} = 1$. Moreover, the self-financing condition must be satisfied

$$\alpha_{n,l-1} N_{l-1} \cdot S(\tau_{n,l}) = \alpha_{n,l} N_l \cdot S(\tau_{n,l}).$$

The above relationship fully defines $\alpha_{n,l}$ as an $\mathcal{F}_{\tau_{n,l}}$-measurable random variable.

Now, for arbitrary $\delta, \epsilon > 0$ we will find $N(\delta,\epsilon)$ such that for $n \ge N(\delta,\epsilon)$ there exists a set $A_n(\delta,\epsilon) \subseteq \Omega$ with $\mathbb{P}^y\{A_n(\delta,\epsilon)\} \ge 1 - \epsilon$ and

$$A_n(\delta,\epsilon) \subseteq \Big\{\frac{S^i(\tau_{n,l})}{S^i(\tau_l)} \in [e^{-\delta}, e^\delta], \quad i = 1,\dots,d, \quad l = 0,\dots,m\Big\}. \tag{19}$$

By Lemma 6.8 there exists a compact set $\tilde{B}_1 \subseteq \mathbb{R}^d \times \mathbb{R}^m$ such that $y \in \tilde{B}_1$ and

$$\mathbb{P}^y\big\{\big(Z(t), X(t)\big) \notin \tilde{B}_1 \quad \text{for } t \in [0,T]\big\} < \frac{\epsilon}{m+1}.$$

From Corollary 6.10 we can find an $N \in \mathbb{N}$ such that for $n \geq N$ the n-discretization Π_n of Π satisfies

$$\mathbb{P}^y\big\{|Z(\tau_{n,l}) - Z(\tau_l)| > \delta, \quad \big(Z(\tau_l), X(\tau_l)\big) \in \tilde{B}_1\big\} \leq \frac{\epsilon}{m+1}, \quad l = 1, \ldots, m.$$

Therefore,

$$\begin{aligned}
\mathbb{P}^y\{A_n^c(\delta,\epsilon)\} &= \mathbb{P}^y\big\{\exists_{l\in\{1,2,\ldots,m\}} |Z(\tau_{n,l}) - Z(\tau_l)| > \delta\big\} \\
&= \mathbb{P}^y\big\{\exists_{l\in\{1,2,\ldots,m\}} |Z(\tau_{n,l}) - Z(\tau_l)| > \delta, \quad \exists_{t\in[0,T]} \big(Z(\tau_l), X(\tau_l)\big) \notin \tilde{B}_1\big\} \\
&\quad + \mathbb{P}^y\big\{\exists_{l\in\{1,2,\ldots,m\}} |Z(\tau_{n,l}) - Z(\tau_l)| > \delta, \quad \forall_{t\in[0,T]} \big(Z(\tau_l), X(\tau_l)\big) \in \tilde{B}_1\big\} \\
&\leq \mathbb{P}^y\big\{\exists_{t\in[0,T]} \big(Z(\tau_l), X(\tau_l)\big) \notin \tilde{B}_1\big\} \\
&\quad + \sum_{l=1}^m \mathbb{P}^y\big\{|Z(\tau_{n,l}) - Z(\tau_l)| > \delta, \quad \forall_{t\in[0,T]} \big(Z(\tau_l), X(\tau_l)\big) \in \tilde{B}_1\big\} \\
&\leq \frac{\epsilon}{m+1} + m\frac{\epsilon}{m+1} = \epsilon.
\end{aligned}$$

Fix $\delta, \epsilon > 0$ and take $n \geq N(\delta,\epsilon)$. We will provide an estimate of α_m from below on the set $A_n(\delta,\epsilon)$. Fix a transaction number $l \in \{1,\ldots,m\}$. The wealth before the transaction is equal to $N_{n,l-1} \cdot S(\tau_{n,l})$. Relation (19) allows us to make the following sequence of statements:

$$\begin{aligned}
N_{n,l-1} \cdot S(\tau_{n,l}) = \alpha_{n,l-1} N_{l-1} \cdot S(\tau_{n,l}) &= \alpha_{n,l-1} \sum_{i=1}^d N_{l-1}^i S^i(\tau_{n,l}) \\
&\geq \alpha_{n,l-1} \sum_{i=1}^d N_{l-1}^i S^i(\tau_l') e^{-\delta} = \alpha_{n,l-1} e^{-\delta} N_{l-1} \cdot S(\tau_l).
\end{aligned}$$

Since Π is self-financing

$$\alpha_{l-1} e^{-\delta} N_{l-1} \cdot S(\tau_l) = \alpha_{l-1} e^{-\delta} N_l \cdot S(\tau_l).$$

Consequently, by (19)

$$\alpha_{l-1} e^{-\delta} N_l \cdot S(\tau_l) = \alpha_{l-1} e^{-\delta} \sum_{i=1}^d N_l^i S^i(\tau_l) \geq \alpha_{l-1} e^{-\delta} \sum_{i=1}^d N_l^i S^i(\tau_l') e^{-\delta} = \alpha_{l-1} e^{-2\delta} N_l \cdot S(\tau_l').$$

Therefore, we have obtained the lower bound for α_l in the recurrent form: $\alpha_l \geq \alpha_{l-1} e^{-2\delta}$. Hence,

$$\alpha_m \geq e^{-2m\delta}.$$

Consequently, we obtain the estimate

$$\begin{aligned}
\mathbb{E}^y\big(1_{A_n(\delta,\epsilon)} N_{n,m} \cdot S(T)\big)^2 = \mathbb{E}^y\big(1_{A_n(\delta,\epsilon)} \alpha_m N_m \cdot S(T)\big)^2 &\geq \mathbb{E}^y\big(1_{A_n(\delta,\epsilon)} e^{-2m\delta} N_m \cdot S(T)\big)^2 \\
&= e^{-4m\delta} \mathbb{E}^y\big(1_{A_n(\delta,\epsilon)} N_m \cdot S(T)\big)^2.
\end{aligned}$$

We come to the final step of the proof. We easily obtain the following estimate

$$\begin{aligned}
\mathbb{E}^y\big(N_m \cdot S(T)\big)^2 &= \mathbb{E}^y\big(1_{A_n(\delta,\epsilon)} N_m \cdot S(T)\big)^2 + \mathbb{E}^y\big(1_{A_n^c(\delta,\epsilon)} N_m \cdot S(T)\big)^2 \\
&\leq e^{4m\delta} \mathbb{E}^y\big(1_{A_n(\delta,\epsilon)} N_{n,m} \cdot S(T)\big)^2 + \mathbb{E}^y\big(1_{A_n^c(\delta,\epsilon)} N_m \cdot S(T)\big)^2 \\
&\leq e^{4m\delta} \mathbb{E}^y\big(N_{n,m} \cdot S(T)\big)^2 + \mathbb{E}^y\big(1_{A_n^c(\delta,\epsilon)} N_m \cdot S(T)\big)^2 \\
&\leq e^{4m\delta} \lim_{k\to\infty} \sup_{\Pi' \in \tilde{\mathcal{A}}_k^m(y)} \mathbb{E}^y\big(N^{\Pi'}(T) \cdot S(T)\big)^2 + \mathbb{E}^y\big(1_{A_n^c(\delta,\epsilon)} N_m \cdot S(T)\big)^2. \quad (20)
\end{aligned}$$

Observe that (20) depends on n only by means of $A_n(\delta,\epsilon)$. Let $\delta=\epsilon$ converge to zero. Then

$$e^{4m\delta}\lim_{k\to\infty}\sup_{\Pi'\in\tilde{\mathcal{A}}^m_k(y)}\mathbb{E}^{\,y}\big(N^{\Pi'}(T)\cdot S(T)\big)^2\to\lim_{k\to\infty}\sup_{\Pi'\in\tilde{\mathcal{A}}^m_k(y)}\mathbb{E}^{\,y}\big(N^{\Pi'}(T)\cdot S(T)\big)^2$$

and

$$\mathbb{E}^{\,y}\big(1_{A^c_n(\delta,\epsilon)}N_m\cdot S(T)\big)^2\to 0$$

by the bounded convergence theorem, since Lemma 6.6 assures that $N_m\cdot S(T)$ is square integrable.■

Define

$$W(m,y)=\sup_{\Pi\in\tilde{\mathcal{A}}^m(y)}\mathbb{E}^{\,y}\big(N^{\Pi}(T)\cdot S(T)\big)^2,\quad m\in\mathbb{N},\quad y\in E.$$

For a strategy $\Pi\in\tilde{\mathcal{A}}^m(y)$, $y=(\eta_0,s_0,x_0)$, by Lemma 6.11 and 6.7

$$\mathbb{E}^{\,y}\big(N^{\Pi}(T)\cdot S(T)\big)^2\le\lim_{n\to\infty}\sup_{\Pi'\in\tilde{\mathcal{A}}^m_n(y)}\mathbb{E}^{\,y}\big(N^{\Pi'}(T)\cdot S(T)\big)^2\le(\eta_0\cdot s_0)^2\,e^{2\beta T}.$$

Hence, the sequence $\big(W(m,y)\big)_{m\in\mathbb{N}}$ is bounded by $(\eta_0\cdot s_0)^2\,e^{2\beta T}$ for $y=(\eta_0,s_0,x_0)$. Moreover, it is non-decreasing of m and converges to

$$\sup_{\Pi\in\mathcal{A}(y)}\mathbb{E}^{\,y}\big(N^{\Pi}(T)\cdot S(T)\big)^2.$$

■

6b. Diffusion with bounded coefficients depending on Markovian risk factors

We model the price process as a Doleans-Dade exponential of a diffusion with coefficients depending on an external process representing risk factors:

$$\frac{dS^i(t)}{S^i(t)}=\mu^i\big(X(t)\big)dt+\sigma^i\big(X(t)\big)\cdot dW(t),\quad i=1,\dots,d, \tag{21}$$

where $W(t)$ is a p-dimensional Brownian motion, $\mu^i:\mathbb{R}^m\to\mathbb{R}$ is a drift, $\sigma^i:\mathbb{R}^m\to\mathbb{R}^p$ is a volatility vector and $X(t)$ is a Feller process. We assume that μ^i and σ^i are bounded functions. The solution to (21) can be written explicitly

$$S^i(t)=S^i(s)\exp\bigg(\int_s^t\sigma^i\big(X(u)\big)\cdot dW(u)-\frac{1}{2}\int_s^t\|\sigma^i\big(X(u)\big)\|^2du+\int_s^t\mu^i\big(X(u)\big)du\bigg),$$
$$i=1,\dots,d.$$

The process $S(t)$ satisfies condition MLT of the previous example with

$$\beta=\sup_{i=1,\dots,d}\big(\frac{1}{2}\sup_{x\in\mathbb{R}^m}\|\sigma^i(x)\|^2+\sup_{x\in\mathbb{R}^m}\mu^i(x)\big). \tag{22}$$

However, thanks to the special form of $S(t)$ we will be able to obtain stronger results than in the previous example, especially we will cover the case of unbounded reward function of consumption G and we will lower the bound for the discount factor α.

LEMMA 6.12. For $y = (\eta, s, x) \in E$, $\Pi \in \mathcal{A}(y)$, $t \geq 0$,

$$\mathbb{E}^{\,y}\big\{N^{\Pi}(t) \cdot S(t)\big\} \leq \eta \cdot s\; e^{\tilde{\beta} t}, \tag{23}$$

where

$$\tilde{\beta} = \sup_{(x,i) \in \mathbb{R}^m \times \{1,\ldots,d\}} \mu^i(x).$$

Proof. Fix y, Π and t. For simplicity of the notation we will skip the superscript Π in N^{Π} and τ_k^{Π}. We shall show that for any $i \in \mathbb{N}$

$$\mathbb{E}^{\,y}\big\{N(\tau_i \wedge t) \cdot S(\tau_i \wedge t) e^{\tilde{\beta}(t - \tau_i \wedge t)}\big\} \leq \eta \cdot s\; e^{\beta t}.$$

The self-financing condition implies

$$N(\tau_i \wedge t) \cdot S(\tau_i \wedge t) \leq N(\tau_{i-1} \wedge t) \cdot S(\tau_i \wedge t).$$

Moreover, from the strong Markov property of Feller processes it results that

$$\mathbb{E}^{\,y}\big\{N(\tau_i \wedge t) \cdot S(\tau_i \wedge t) e^{\tilde{\beta}(t - \tau_i \wedge t)}\big\} \leq \mathbb{E}^{\,y}\Big\{\mathbb{E}^{\,\Pi}\big\{N(\tau_{i-1} \wedge t) \cdot S(\tau_i \wedge t) e^{\tilde{\beta}(t - \tau_i \wedge t)}\big|\mathcal{F}_{\tau_{i-1} \wedge t}\big\}\Big\}$$

We expand the scalar product under the conditional expectation

$$\begin{aligned}
&\mathbb{E}^{\,y}\big\{N(\tau_{i-1} \wedge t) \cdot S(\tau_i \wedge t) e^{\tilde{\beta}(t - \tau_i \wedge t)}\big|\mathcal{F}_{\tau_{i-1} \wedge t}\big\} \\
&\qquad\qquad = \sum_{k=1}^{d} \mathbb{E}^{\,y}\big\{N^k(\tau_{i-1} \wedge t) S^k(\tau_i \wedge t) e^{\tilde{\beta}(t - \tau_i \wedge t)}\big|\mathcal{F}_{\tau_{i-1} \wedge t}\big\}
\end{aligned}$$

We estimate each term separately

$$\begin{aligned}
&\mathbb{E}^{\,y}\big\{N^k(\tau_{i-1} \wedge t) S^k(\tau_i \wedge t) e^{\beta(t - \tau_i \wedge t)}\big|\mathcal{F}_{\tau_{i-1} \wedge t}\big\} \\
&\quad = N^k(\tau_{i-1} \wedge t)\; \mathbb{E}^{\,y}\Big\{e^{\tilde{\beta}(t - \tau_i \wedge t)} S^k(\tau_{i-1} \wedge t) \\
&\qquad\qquad \exp\bigg(\int_{\tau_{i-1} \wedge t}^{\tau_i \wedge t} \sigma^k\big(X(u)\big) dW(u) - \frac{1}{2}\int_{\tau_{i-1} \wedge t}^{\tau_i \wedge t} \big\|\sigma^k\big(X(u)\big)\big\|^2 du \\
&\qquad\qquad\qquad + \int_{\tau_{i-1} \wedge t}^{\tau_i \wedge t} \mu^k\big(X(u)\big) du\bigg)\bigg|\mathcal{F}_{\tau_{i-1} \wedge t}\Big\} \\
&\quad \leq N^k(\tau_{i-1} \wedge t) S^k(\tau_{i-1} \wedge t) \\
&\qquad \mathbb{E}^{\,\Pi}\Big\{e^{\tilde{\beta}(t - \tau_i \wedge t)} e^{\tilde{\beta}(\tau_i \wedge t - \tau_{i-1} \wedge t)} \exp\bigg(\int_{\tau_{i-1} \wedge t}^{\tau_i \wedge t} \sigma^i\big(X(u)\big) dW(u) \\
&\qquad\qquad\qquad - \frac{1}{2}\int_{\tau_{i-1} \wedge t}^{\tau_i \wedge t} \big\|\sigma^i\big(X(u)\big)\big\|^2 du\bigg)\bigg|\mathcal{F}_{\tau_{i-1} \wedge t}\Big\} \\
&\quad = N^k(\tau_{i-1} \wedge t) S^k(\tau_{i-1} \wedge t) e^{\tilde{\beta}(\tau_{i-1} \wedge t)},
\end{aligned}$$

since

$$s \mapsto \exp\bigg(\int_0^s \sigma^i\big(X(u)\big) dW(u) - \frac{1}{2}\int_0^s \big\|\sigma^i\big(X(u)\big)\big\|^2 du\bigg)$$

is a martingale. To sum it up, we have just shown that

$$\mathbb{E}^{\,y}\big\{N(\tau_i\wedge t)\cdot S(\tau_i\wedge t)e^{\tilde\beta(t-\tau_i\wedge t)}\big\}\le \mathbb{E}^{\,y}N(\tau_{i-1}\wedge t)\cdot S(\tau_{i-1}\wedge t)e^{\tilde\beta(\tau_{i-1}\wedge t)}.$$

Repeating the above reasoning $i-1$ times we obtain

$$\mathbb{E}^{\,y}\big\{N(\tau_i\wedge t)\cdot S(\tau_i\wedge t)e^{\tilde\beta(t-\tau_i\wedge t)}\big\}\le \eta\cdot s\; e^{\tilde\beta t}.$$

To finish the proof we let i go to infinity. By the Fatou's lemma

$$\mathbb{E}^{\,y}\liminf_{i\to\infty}\big\{N(\tau_i\wedge t)\cdot S(\tau_i\wedge t)e^{\tilde\beta(t-\tau_i\wedge t)}\big\}\le \eta\cdot s\; e^{\tilde\beta t}.$$

The strategy Π is admissible, so $\tau_i(\omega)>\sigma(\omega)$ for infinitely many i. Hence

$$\liminf_{k\to\infty}\big\{N(\tau_i\wedge t)\cdot S(\tau_i\wedge t)e^{\tilde\beta(t-\tau_i\wedge t)}\big\}=N(t)\cdot S(t),$$

which completes the proof. ∎

Observe that by Lemma 6.12 NFL is clearly satisfied. We can prove NFL2 in a particular situation.

COROLLARY 6.13. If $F(\eta,s,x)=(\eta\cdot s)^\gamma$ for $\gamma\in(0,1)$, $G\equiv 0$ and $\alpha>\tilde\beta$, the condition NFL2 is satisfied.

Proof. Fix a compact set $B\subseteq E$ and $\kappa=\frac{1}{\gamma}$. Take $y=(\eta,s,x)\in B$ and $\Pi\in\mathcal{A}(y)$. By Lemma 6.12

$$\begin{aligned}\mathbb{E}^{\,y}\int_0^\infty e^{-\alpha t}\big(N^\Pi(t)\cdot S(t)\big)^{\gamma\kappa}dt&=\mathbb{E}^{\,y}\int_0^\infty e^{-\alpha t}N^\Pi(t)\cdot S(t)dt\\&=\int_0^\infty e^{-\alpha t}\mathbb{E}^{\,y}\big\{N^\Pi(t)\cdot S(t)\big\}dt\\&\le\int_0^\infty e^{-\alpha t}\eta\cdot s\; e^{\tilde\beta t}dt=\frac{\eta\cdot s}{\alpha-\tilde\beta}.\end{aligned}$$

Hence NFL2 is satisfied as $\eta\cdot s$ is bounded on compacts. ∎

To obtain results for a wider family of reward functions F and $G\ge 0$, we need a more sophisticated version of Lemma 6.12. Observe that in Lemma 6.14 we derive the estimate for a random time σ whereas in the previous subsection we could only prove a similar result for deterministic times (see Corollary 6.2). This result is crucial for the extension of the class of tractable functions G. Notice that $\hat\beta$ defined below is different from β in (22).

LEMMA 6.14. For $y=(\eta,s,x)\in E$, $\Pi\in\mathcal{A}(y)$, $\gamma\ge 0$ and a bounded stopping time σ

$$\mathbb{E}^{\,y}\big\{e^{-\gamma\sigma}N^\Pi(\sigma)\cdot S(\sigma)\big\}^2\le(\eta\cdot s)^2\Big(\mathbb{E}^{\,y}\big\{e^{-8(\hat\beta-\gamma)\sigma}\big\}\Big)^{1/4},\tag{24}$$

where

$$\hat\beta=\sup_{i=1,\dots,d}\big(\sup_{x\in\mathbb{R}^m}\|\sigma^i(x)\|^2+\sup_{x\in\mathbb{R}^m}\mu^i(x)\big).$$

Proof. Similarly as before we shall skip the superscript Π in N^{Π} and τ_i^{Π}. Fix $y = (\eta, s, x) \in E$ and $\Pi \in \mathcal{A}(y)$. First we will show that for arbitrary $i \in \mathbb{N}$ and a bounded non-negative random variable ξ

$$\begin{aligned}
&\mathbb{E}^{\,y}\Big\{ \Big(e^{-\gamma(\tau_i \wedge \sigma)} N(\tau_i \wedge \sigma) \cdot S(\tau_i \wedge \sigma) \Big)^2 \Big(\mathbb{E}^{\,y}\big\{ \xi | \mathcal{F}_{\tau_i \wedge \sigma} \big\} \Big)^{1/4} \Big\} \\
&\quad \leq \mathbb{E}^{\,\Pi}\Big\{ \Big(e^{-\gamma(\tau_{i-1} \wedge \sigma)} N(\tau_{i-1} \wedge \sigma) \cdot S(\tau_{i-1} \wedge \sigma) \Big)^2 \Big(\mathbb{E}^{\,y}\big\{ \xi e^{8(\hat{\beta}-\gamma)(\tau_i \wedge \sigma - \tau_{i-1} \wedge \sigma)} \big| \mathcal{F}_{\tau_{i-1} \wedge \sigma} \big\} \Big)^{1/4} \Big\}.
\end{aligned} \tag{25}$$

The self-financing condition implies

$$N(\tau_i \wedge t) \cdot S(\tau_i \wedge t) \leq N(\tau_{i-1} \wedge t) \cdot S(\tau_i \wedge t).$$

Consequently,

$$\begin{aligned}
&\mathbb{E}^{\,y}\Big\{ \Big(e^{-\gamma(\tau_i \wedge \sigma)} N(\tau_i \wedge \sigma) \cdot S(\tau_i \wedge \sigma) \Big)^2 \Big(\mathbb{E}^{\,y}\big\{ \xi | \mathcal{F}_{\tau_i \wedge \sigma} \big\} \Big)^{1/4} \Big\} \\
&\qquad \leq \mathbb{E}^{\,y}\Big\{ \mathbb{E}^{\,y}\Big\{ \Big(e^{-\gamma(\tau_i \wedge \sigma)} N(\tau_{i-1} \wedge \sigma) \cdot S(\tau_i \wedge \sigma) \Big)^2 \Big(\mathbb{E}^{\,y}\big\{ \xi | \mathcal{F}_{\tau_i \wedge \sigma} \big\} \Big)^{1/4} \Big| \mathcal{F}_{\tau_{i-1} \wedge \sigma} \Big\} \Big\}.
\end{aligned}$$

We expand the scalar product under the conditional expectation

$$\begin{aligned}
&\mathbb{E}^{\,y}\Big\{ \Big(e^{-\gamma(\tau_i \wedge \sigma)} N(\tau_{i-1} \wedge \sigma) \cdot S(\tau_i \wedge \sigma) \Big)^2 \Big(\mathbb{E}^{\,y}\big\{ \xi | \mathcal{F}_{\tau_i \wedge \sigma} \big\} \Big)^{1/2} \Big| \mathcal{F}_{\tau_{i-1} \wedge \sigma} \Big\} \\
&\qquad = \sum_{k,l=1}^{d} N^k(\tau_{i-1} \wedge \sigma) N^l(\tau_{i-1} \wedge \sigma) \\
&\qquad\qquad \mathbb{E}^{\,y}\Big\{ e^{-2\gamma(\tau_i \wedge \sigma)} S^k(\tau_i \wedge \sigma) S^l(\tau_i \wedge \sigma) \Big(\mathbb{E}^{\,y}\big\{ \xi | \mathcal{F}_{\tau_i \wedge \sigma} \big\} \Big)^{1/4} \Big| \mathcal{F}_{\tau_{i-1} \wedge \sigma} \Big\}
\end{aligned}$$

and estimate each term separately

$$\begin{aligned}
&N^k(\tau_{i-1} \wedge \sigma) N^l(\tau_{i-1} \wedge \sigma) \mathbb{E}^{\,y}\Big\{ e^{-2\gamma(\tau_i \wedge \sigma)} S^k(\tau_i \wedge \sigma) S^l(\tau_i \wedge \sigma) \Big(\mathbb{E}^{\,y}\big\{ \xi | \mathcal{F}_{\tau_i \wedge \sigma} \big\} \Big)^{1/4} \Big| \mathcal{F}_{\tau_{i-1} \wedge \sigma} \Big\} \\
&= e^{-2\gamma(\tau_{i-1} \wedge \sigma)} N^k(\tau_{i-1} \wedge \sigma) N^l(\tau_{i-1} \wedge \sigma) S^k(\tau_{i-1} \wedge \sigma) S^l(\tau_{i-1} \wedge \sigma) \\
&\quad \mathbb{E}^{\,y}\Big\{ \exp\Big(\int_{\tau_{i-1} \wedge \sigma}^{\tau_i \wedge \sigma} \sigma^k\big(X(u)\big) dW(u) - \frac{1}{2} \int_{\tau_{i-1} \wedge \sigma}^{\tau_i \wedge \sigma} \big\| \sigma^k\big(X(u)\big) \big\|^2 du + \int_{\tau_{i-1} \wedge \sigma}^{\tau_i \wedge \sigma} \mu^k\big(X(u)\big) du \Big) \\
&\qquad \exp\Big(\int_{\tau_{i-1} \wedge \sigma}^{\tau_i \wedge \sigma} \sigma^l\big(X(u)\big) dW(u) - \frac{1}{2} \int_{\tau_{i-1} \wedge \sigma}^{\tau_i \wedge \sigma} \big\| \sigma^l\big(X(u)\big) \big\|^2 du + \int_{\tau_{i-1} \wedge \sigma}^{\tau_i \wedge \sigma} \mu^l\big(X(u)\big) du \Big) \\
&\qquad\qquad e^{-2\gamma(\tau_i \wedge \sigma - \tau_{i-1} \wedge \sigma)} \Big(\mathbb{E}^{\,y}\big\{ \xi | \mathcal{F}_{\tau_i \wedge \sigma} \big\} \Big)^{1/4} \Big| \mathcal{F}_{\tau_{i-1} \wedge \sigma} \Big\}.
\end{aligned}$$

By Schwartz inequality and the martingale property of Doleans-Dade exponential

$$
\begin{aligned}
&\mathbb{E}^{\,y}\Big\{ \exp\Big(\int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \sigma^k\big(X(u)\big)dW(u) - \frac{1}{2}\int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \big\|\sigma^k\big(X(u)\big)\big\|^2 du + \int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \mu^k\big(X(u)\big)du \Big) \\
&\qquad \exp\Big(\int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \sigma^l\big(X(u)\big)dW(u) - \frac{1}{2}\int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \big\|\sigma^l\big(X(u)\big)\big\|^2 du + \int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \mu^l\big(X(u)\big)du \Big) \\
&\qquad\qquad e^{-2\gamma(\tau_i\wedge\sigma-\tau_{i-1}\wedge\sigma)}\Big(\mathbb{E}^{\,y}\big\{\xi|\mathcal{F}_{\tau_i\wedge\sigma}\big\}\Big)^{1/4}\Big|\mathcal{F}_{\tau_{i-1}\wedge\sigma}\Big\} \\
&\leq \Big(\mathbb{E}^{\,y}\Big\{ \exp\Big(\int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} 2\sigma^l\big(X(u)\big)dW(u) - \int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \big\|\sigma^l\big(X(u)\big)\big\|^2 du + \int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} 2\mu^l\big(X(u)\big)du \Big) \\
&\qquad \exp\Big(\int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \big\|\sigma^k\big(X(u)\big)\big\|^2 du + \int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} 2\mu^k\big(X(u)\big)du - 4\gamma(\tau_i\wedge\sigma-\tau_{i-1}\wedge\sigma)\Big) \\
&\qquad\qquad \Big(\mathbb{E}^{\,y}\big\{\xi|\mathcal{F}_{\tau_i\wedge\sigma}\big\}\Big)^{1/2}\Big|\mathcal{F}_{\tau_{i-1}\wedge\sigma}\Big\}\Big)^{1/2} \\
&\leq \Big(\mathbb{E}^{\,y}\Big\{ \exp\Big(6\int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \big\|\sigma^l\big(X(u)\big)\big\|^2 du + \int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} 4\mu^l\big(X(u)\big)du \Big) \\
&\qquad \exp\Big(2\int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} \big\|\sigma^k\big(X(u)\big)\big\|^2 du + \int_{\tau_{i-1}\wedge\sigma}^{\tau_i\wedge\sigma} 4\mu^k\big(X(u)\big)du - 8\gamma(\tau_i\wedge\sigma-\tau_{i-1}\wedge\sigma)\Big) \\
&\qquad\qquad \mathbb{E}^{\,y}\big\{\xi|\mathcal{F}_{\tau_i\wedge\sigma}\big\}\Big|\mathcal{F}_{\tau_{i-1}\wedge\sigma}\Big\}\Big)^{1/4} \\
&\leq \Big(\mathbb{E}^{\,y}\big\{e^{8(\hat{\beta}-\gamma)(\tau_i\wedge\sigma-\tau_{i-1}\wedge\sigma)}\xi\big|\mathcal{F}_{\tau_{i-1}\wedge\sigma}\big\}\Big)^{1/4}.
\end{aligned}
$$

Therefore, we easily obtain (25).

Now, we prove that for any $i \in \mathbb{N}$

$$
\mathbb{E}^{\,y}\Big\{e^{-\gamma(\tau_i\wedge\sigma)}N(\tau_i\wedge\sigma)\cdot S(\tau_i\wedge\sigma)\Big\}^2 \leq (\eta\cdot s)^2\Big(\mathbb{E}^{\,y}\big\{e^{-8(\hat{\beta}-\gamma)\sigma}\big\}\Big)^{1/4}. \tag{26}
$$

Using (25) with $\xi = 0$

$$
\begin{aligned}
&\mathbb{E}^{\,y}\Big\{e^{-\gamma(\tau_i\wedge\sigma)}N(\tau_i\wedge\sigma)\cdot S(\tau_i\wedge\sigma)\Big\}^2 \\
&\quad\leq \mathbb{E}^{\,y}\Big\{\Big(e^{-\gamma(\tau_{i-1}\wedge\sigma)}N(\tau_{i-1}\wedge\sigma)\cdot S(\tau_{i-1}\wedge\sigma)\Big)^2\Big(\mathbb{E}^{\,y}\big\{e^{8(\hat{\beta}-\gamma)(\tau_i\wedge\sigma-\tau_{i-1}\wedge\sigma)}\big|\mathcal{F}_{\tau_{i-1}\wedge\sigma}\big\}\Big)^{1/4}\Big\}.
\end{aligned}
$$

Next, we apply (25) with $\xi = \exp\big(8(\hat{\beta}-\gamma)(\tau_i\wedge\sigma-\tau_{i-1}\wedge\sigma)\big)$

$$
\begin{aligned}
&\mathbb{E}^{\,y}\Big\{\Big(e^{-\alpha(\tau_{i-1}\wedge\sigma)}N(\tau_{i-1}\wedge\sigma)\cdot S(\tau_{i-1}\wedge\sigma)\Big)^2\Big(\mathbb{E}^{\,y}\big\{e^{8(\hat{\beta}-\gamma)(\tau_i\wedge\sigma-\tau_{i-1}\wedge\sigma)}\big|\mathcal{F}_{\tau_{i-1}\wedge\sigma}\big\}\Big)^{1/4}\Big\} \\
&\quad\leq \mathbb{E}^{\,y}\Big\{\Big(e^{-\alpha(\tau_{i-2}\wedge\sigma)}N(\tau_{i-2}\wedge\sigma)\cdot S(\tau_{i-2}\wedge\sigma)\Big)^2\Big(\mathbb{E}^{\,y}\big\{e^{8(\hat{\beta}-\gamma)(\tau_i\wedge\sigma-\tau_{i-2}\wedge\sigma)}\big|\mathcal{F}_{\tau_{i-2}\wedge\sigma}\big\}\Big)^{1/4}\Big\}.
\end{aligned}
$$

We repeat the last step $i-2$ times and obtain (26). We send i to infinity. The monotone convergence theorem and the admissibility of Π yield

$$
\lim_{i\to\infty}\mathbb{E}^{\,y}\big\{e^{8(\hat{\beta}-\gamma)(\tau_i\wedge\sigma)}\big\} = \mathbb{E}^{\,y}\big\{e^{8(\hat{\beta}-\gamma)\sigma}\big\}.
$$

Since Π is admissible, $\tau_i(\omega) > \sigma(\omega)$ apart from finitely many i. So by Fatou's lemma

$$\liminf_{i\to\infty} \mathbb{E}^{\,y}\Big\{e^{-\gamma(\tau_i\wedge\sigma)}N(\tau_i\wedge\sigma)\cdot S(\tau_i\wedge\sigma)\Big\}^2 \geq \mathbb{E}^{\,y}\Big\{e^{-\gamma\sigma}N(\sigma)\cdot S(\tau_i\wedge\sigma)\Big\}^2.$$

Therefore,

$$\mathbb{E}^{\,y}\Big\{e^{-\gamma\sigma}N(\sigma)\cdot S(\tau_i\wedge\sigma)\Big\}^2 \leq (\eta\cdot s)^2\,\left(\mathbb{E}^{\,y}\big\{e^{8(\hat{\beta}-\gamma)\sigma}\big\}\right)^{1/4}.$$

■

COROLLARY 6.15. If $\gamma \geq \hat{\beta}$ then the assumption on boundedness of the stopping time σ in Lemma 6.14 can be skipped, i.e. for $y = (\eta, s, x) \in E$, $\Pi \in \mathcal{A}(y)$ and any stopping time σ

$$\mathbb{E}^{\,y}\big\{e^{-\gamma\sigma}N^{\Pi}(\sigma)\cdot S(\sigma)\big\}^2 \leq (\eta\cdot s)^2\Big(\mathbb{E}^{\,y}\big\{e^{-8(\gamma-\hat{\beta})\sigma}\big\}\Big)^{1/4}, \tag{27}$$

Proof. By Lemma 6.14 for any $T > 0$

$$\mathbb{E}^{\,y}\Big\{e^{-\gamma(\sigma\wedge T)}N^{\Pi}(\sigma\wedge T)\cdot S(\sigma\wedge T)\Big\}^2 \leq (\eta\cdot s)^2\,\left(\mathbb{E}^{\,y}\Big\{e^{8(\sigma\wedge T)(\hat{\beta}-\gamma)}\Big\}\right)^{1/4}.$$

Since $(\hat{\beta}-\gamma) \leq 0$, monotone convergence theorem yields

$$\lim_{T\to\infty}\left(\mathbb{E}^{\,y}\Big\{e^{8(\sigma\wedge T)(\hat{\beta}-\gamma)}\Big\}\right)^{1/4} = \left(\mathbb{E}^{\,y}\Big\{e^{8\sigma(\hat{\beta}-\gamma)}\Big\}\right)^{1/4}$$

and the limit is finite. Fatou's lemma implies

$$\begin{aligned}
&\mathbb{E}^{\,y}\liminf_{T\to\infty}\Big\{e^{-\gamma(\sigma\wedge T)}N^{\Pi}(\sigma\wedge T)\cdot S(\sigma\wedge T)\Big\}^2\\
&\leq \liminf_{T\to\infty}\mathbb{E}^{\,y}\Big\{e^{-\gamma(\sigma\wedge T)}N^{\Pi}(\sigma\wedge T)\cdot S(\sigma\wedge T)\Big\}^2\\
&\leq (\eta\cdot s)^2\,\left(\mathbb{E}^{\,y}\Big\{e^{8\sigma(\hat{\beta}-\gamma)}\Big\}\right)^{1/4}.
\end{aligned}$$

Obviously

$$\liminf_{T\to\infty}\Big(e^{-\gamma(\sigma\wedge T)}N^{\Pi}(\sigma\wedge T)\cdot S(\sigma\wedge T)\Big) = e^{-\gamma\sigma}N^{\Pi}(\sigma)\cdot S(\sigma),$$

which completes the proof. ■

COROLLARY 6.16. If $\hat{\beta} < 2\alpha$, $\kappa \in [1, \frac{2\alpha}{\hat{\beta}} \wedge 2]$ then for $y = (\eta, s, x) \in E$, $\Pi \in \mathcal{A}(y)$ and any stopping time σ

$$\mathbb{E}^{\,y}\big\{e^{-\alpha\sigma}\big(N^{\Pi}(\sigma)\cdot S(\sigma)\big)^{\kappa}\big\} \leq (\eta\cdot s)^{\kappa}\Big(\mathbb{E}^{\,y}\big\{e^{-8(\frac{2\alpha}{\kappa}-\hat{\beta})\sigma}\big\}\Big)^{\kappa/8}, \tag{28}$$

and consequently

$$\mathbb{E}^{\,y}\big\{e^{-\alpha\sigma}\big(N^{\Pi}(\sigma)\cdot S(\sigma)\big)^{\kappa}\big\} \leq (\eta\cdot s)^{\kappa}.$$

Proof. It is an easy application of Hölder inequality and Corollary 6.15. ■

COROLLARY 6.17. If $\hat{\beta} < 2\alpha$, $\kappa \in [1, \frac{2\alpha}{\hat{\beta}} \wedge 2]$ then for $y \in E$, $\Pi \in \mathcal{A}(y)$ and $l < k$

$$\mathbb{E}^{\,y}\big\{e^{-\alpha\tau_k^\Pi}\big(N^\Pi(\tau_k^\Pi)\cdot S(\tau_k^\Pi)\big)^\kappa\big|\mathcal{F}_{\tau_l^\Pi}\big\} \le e^{-\alpha\tau_l^\Pi}\big(N^\Pi(\tau_l^\Pi)\cdot S(\tau_l^\Pi)\big)^\kappa.$$

Proof. It follows from the proof of Lemma 6.14 and the arguments used in the above corollaries. ■

We shall prove the existence of an optimal strategy for reward functions F satisfying assumption AF, i.e.

$$\exists A, B \ge 0 \qquad 0 \le F(\eta, s, x) \le A + B(\eta\cdot s), \quad (\eta, s, x) \in E,$$

and reward functions G satisfying

Assumption AG. There exists $A, B \ge 0$ such that

$$G(s) \le A + Bs, \quad s \in \mathbb{R}_+.$$

Observe that without loss of generality the constants in AF and AG can be the same. Notice that condition AG is satisfied by all non-negative HARA utility functions.

We will not use Theorem 5.1. We have to exploit directly the properties of the price process to obtain the result in this generality.

THEOREM 6.18. Assume that F satisfies AF, G satisfies AG, $\alpha > \hat{\beta}/2$,

$$\hat{\beta} = \sup_{i=1,\dots,d}\big(\sup_{x\in\mathbb{R}^m}\|\sigma^i(x)\|^2 + \sup_{x\in\mathbb{R}^m}\mu^i(x)\big),$$

and NUM holds. Then the value function is continuous and satisfies the Bellman equation $v = \mathcal{G}v$. Moreover, there exists an optimal strategy for any $y \in E$.

For the proof of the theorem we shall need the following lemmas:

LEMMA 6.19. Under the assumptions of Theorem 6.18

$$v(\eta, s, x) \le A_1 + B_1(\eta\cdot s), \quad (\eta, s, x) \in E,$$

where $A_1 = \frac{A}{\alpha}$, $B_1 = \frac{B}{2\alpha-\hat{\beta}} + B$ and v is the value function.

Proof. Fix $y = (\eta, s, x) \in E$, $\Pi \in \mathcal{A}(y)$. Recall that

$$J(\Pi) = \mathbb{E}^{\,y}\Big\{\int_0^\infty e^{-\alpha t}F\big(Y^\Pi(t)\big)dt + \sum_{k=1}^\infty e^{-\alpha\tau_k^\Pi}G\big(C_k^\Pi\big)\Big\}. \tag{29}$$

By Corollary 6.16

$$\begin{aligned}\mathbb{E}^{\,y}\int_0^\infty e^{-\alpha t}F\big(Y^\Pi(t)\big)dt &\le A\int_0^\infty e^{-\alpha t}dt + B\int_0^\infty e^{-\alpha t}\mathbb{E}^{\,y}\big(N^\Pi(t)\cdot S(t)\big)dt \\ &\le \frac{A}{\alpha} + B(\eta\cdot s)\int_0^\infty e^{-(2\alpha-\hat{\beta})t}dt = \frac{A}{\alpha} + \frac{B}{2\alpha-\hat{\beta}}(\eta\cdot s).\end{aligned}$$

We shall prove that the second term of (29) satisfies the following inequality:

$$\mathbb{E}^{\,y}\sum_{k=1}^\infty e^{-\alpha\tau_k^\Pi}G(C_k^\Pi) \le B(\eta\cdot s).$$

Obviously, by AG1

$$\mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^{\Pi}} G(C_k^{\Pi}) \le B\mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^{\Pi}} C_k^{\Pi}.$$

By Corollary 6.17 we have

$$\mathbb{E}^{\,y}\Big\{e^{-\alpha\tau_k^{\Pi}} N_{k-1}\cdot S(\tau_k^{\Pi})\big|\mathcal{F}_{\tau_{k-1}^{\Pi}}\Big\} \le e^{-\alpha\tau_{k-1}^{\Pi}} N_{k-1}\cdot S(\tau_{k-1}^{\Pi}).$$

Hence, for any $M \ge 1$

$$\begin{aligned}\mathbb{E}^{\,y}\sum_{k=1}^{M} e^{-\alpha\tau_k^{\Pi}} C_k^{\Pi} &\le \mathbb{E}^{\,y}\Big\{\sum_{k=1}^{M-1} e^{-\alpha\tau_k^{\Pi}} C_k^{\Pi} + \mathbb{E}^{\,y}\Big\{e^{-\alpha\tau_M^{\Pi}}\Big(C_M^{\Pi} + N_M^{\Pi}\cdot S(\tau_M^{\Pi})\Big)\big|\mathcal{F}_{\tau_{M-1}^{\Pi}}\Big\}\Big\}\\ &\le \mathbb{E}^{\,y}\Big\{\sum_{k=1}^{M-1} e^{-\alpha\tau_k^{\Pi}} C_k^{\Pi} + e^{-\alpha\tau_{M-1}^{\Pi}} N_{M-1}^{\Pi}\cdot S(\tau_{M-1}^{\Pi})\Big\}\\ &\le \ldots \le \eta\cdot s.\end{aligned}$$

By monotone convergence theorem

$$\mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^{\Pi}} C_k^{\Pi} \le \eta\cdot s.$$

■

Proof of Theorem 6.18. Recall that the key point in the proof of Theorem 5.1 was to show that functions v_i and v are continuous. Since we cannot prove assumption NFL2 for unbounded G we shall use a different technique. Recall that

$$v_0(y) = \mathbb{E}^{\,y}\int_0^{\infty} e^{-\alpha s} F\big(Y(s)\big)ds$$

and $v_i(y) = \mathcal{G}v_{i-1}(y)$, $i = 1, 2, \ldots$, $y \in E$, where $\mathcal{G}$ is the Bellman operator.

For a continuous function $h : E \to \mathbb{R}_+$ satisfying $h(\eta, s, x) \le A_2 + B_2(\eta\cdot s)$ we define

$$\mathcal{H}^h(y) = \sup_{\tau}\mathbb{E}^{\,y}\Big\{\int_0^{\tau} e^{-\alpha t} F\big(Y(t)\big)dt + e^{-\alpha\tau} h\big(Y(\tau)\big)\Big\}$$

and

$$\mathcal{H}_H^h(y) = \sup_{\tau}\mathbb{E}^{\,y}\Big\{\int_0^{\tau} e^{-\alpha t}\big(F\big(Y(t)\big)\wedge H\big)dt + e^{-\alpha\tau}\big(h\big(Y(\tau)\big)\wedge H\big)\Big\}, \quad H > 0.$$

By the Feller property of $\big(S(t), X(t)\big)$ the value function $\mathcal{H}_H^h$ of the above stopping problem is continuous (see [10]). Observe that

$$\begin{aligned}\mathcal{H}^h(y) - \mathcal{H}_H^h(y) \le &\sup_{\tau}\mathbb{E}^{\,y}\int_0^{\tau} e^{-\alpha s}\big(F\big(Y(s)\big) - H\big)1_{F(Y(s))\ge H}ds\\ &+ \sup_{\tau}\mathbb{E}^{\,y} e^{-\alpha\tau}\big(h\big(Y(\tau)\big) - H\big)1_{h(Y(\tau))\ge H} = (I) + (II).\end{aligned}$$

Fix $H \geq \max(A, A_2)$, $\kappa \in (1, \frac{2\alpha}{\hat{\beta}} \wedge 2)$, $y = (\eta, s, x) \in E$ and observe that

$$(I) \leq \frac{1}{H^{\kappa-1}} \mathbb{E}^{y} \int_0^\infty e^{-\alpha t} \big((F(Y(t)) - A)^+ \big)^\kappa dt \leq \frac{B^\kappa}{H^{\kappa-1}} \int_0^\infty \mathbb{E}^{y} e^{-\alpha t} \big(\eta \cdot S(t)\big)^\kappa dt.$$

Consequently, by Corollary 6.16

$$(I) \leq \frac{B^\kappa}{H^{\kappa-1}} \int_0^\infty e^{-(2\alpha - \kappa\hat{\beta})t} (\eta \cdot s)^\kappa = \frac{B^\kappa (\eta \cdot s)^\kappa}{H^{\kappa-1}(2\alpha - \kappa\hat{\beta})}.$$

Analogously, by Corollary 6.16

$$(II) \leq \frac{B_2^\kappa (\eta \cdot s)^\kappa}{H^{\kappa-1}}.$$

Hence, $\mathcal{H}^h_H$ converges to $\mathcal{H}^h$ uniformly on compact sets and $\mathcal{H}^h$ is a continuous function.

Continuity of v_i is proved by induction. Notice that $v_0 = \mathcal{H}^0$. Fix $k > 0$ and assume that v_k is continuous. Observe that $v_{k+1} = \mathcal{H}^h$, where $h = Mv_k$. By Lemma 3.2 h is a continuous function. Since $v_k \leq v$, using Lemma 6.19 and assumption AG we have $h(\eta, s, x) \leq A + A_1 + (B + B_1)(\eta \cdot s)$. Therefore, by the above argument v_{k+1} is a continuous function. Moreover, by the standard impulsive control argument v_k is a value function for the problem with the number of impulses bounded by k.

We still have to show that v is continuous, i.e. v_k converges to v uniformly on compact sets. Fix a compact set $D \subseteq E$. It suffices to prove that

$$\mathbb{E}^{y} \Big\{ \int_{\tau_i^\Pi}^\infty e^{-\alpha t} F\big(N^\Pi(t) \cdot S(t)\big) dt + \sum_{k=i}^\infty e^{-\alpha \tau_k^\Pi} G(C_k^\Pi) \Big\} \to 0, \quad i \to \infty, \tag{30}$$

uniformly for $y \in D$ and $\Pi \in \mathcal{A}(y)$, since

$$0 \leq v(y) - v_k(y) \leq \mathbb{E}^{y} \Big\{ \int_{\tau_i^\Pi}^\infty e^{-\alpha t} F\big(N^\Pi(t) \cdot S(t)\big) dt + \sum_{k=i}^\infty e^{-\alpha \tau_k^\Pi} G(C_k^\Pi) \Big\}.$$

Uniform convergence of the first term in (30) results from the arguments presented in subsection 6a. However, we shall present a sketch for completeness. Fix $\kappa \in (1, \frac{2\alpha}{\hat{\beta}})$, $y = (\eta, s, x) \in D$ and $\Pi \in \mathcal{A}(y)$. By assumption AF and Hölder inequality

$$\begin{aligned}\mathbb{E}^{y} \Big\{ \int_{\tau_i^\Pi}^\infty e^{-\alpha t} F\big(Y^\Pi(t)\big) dt \Big\} &\leq \mathbb{E}^{y} \Big\{ \int_{\tau_i^\Pi}^\infty e^{-\alpha t} A + \int_{\tau_i^\Pi}^\infty e^{-\alpha t} B\big(N^\Pi(t) \cdot S(t)\big) dt \Big\} \\ &\leq \frac{A}{\alpha} \mathbb{E}^{y} e^{-\alpha \tau_i^\Pi} + B \left\{ \mathbb{E}^{y} \int_0^\infty e^{-\alpha t} \big(N^\Pi(t) \cdot S(t)\big)^\kappa dt \right\}^{\frac{1}{\kappa}} \Big(\frac{1}{\alpha} \mathbb{E}^{y} e^{-\alpha \tau_i^\Pi} \Big)^{\frac{\kappa-1}{\kappa}}\end{aligned}$$

By Corollary 6.16 we obtain the following estimate

$$\frac{A}{\alpha} \mathbb{E}^{y} e^{-\alpha \tau_i^\Pi} + \frac{B(\eta \cdot s)}{(2\alpha - \hat{\beta}\kappa)^{1/\kappa}} \Big(\frac{1}{\alpha} \mathbb{E}^{y} e^{-\alpha \tau_i^\Pi} \Big)^{\frac{\kappa-1}{\kappa}},$$

which by Lemma 4.1 converges to zero uniformly for $\Pi \in \mathcal{A}(y)$, $y \in D$.

Now we shall concentrate on the second term of (30)

$$\mathbb{E}^{y} \sum_{k=i}^\infty e^{-\alpha \tau_k^\Pi} G(C_k^\Pi).$$

Obviously, by AG

$$\mathbb{E}^{\,y}\sum_{k=i}^{\infty}e^{-\alpha\tau_k^\Pi}G(C_k^\Pi)\le A\mathbb{E}^{\,y}\sum_{k=i}^{\infty}e^{-\alpha\tau_k^\Pi}+B\mathbb{E}^{\,y}\sum_{k=i}^{\infty}e^{-\alpha\tau_k^\Pi}C_k^\Pi.$$

First term is uniformly convergent by Lemma 4.3.

Fix $y=(\eta,s,x)\in E$, $\Pi\in\mathcal{A}(y)$. By Corollary 6.17 we have

$$\mathbb{E}^{\,y}\Big\{e^{-\alpha\tau_k^\Pi}N_{k-1}^\Pi\cdot S(\tau_k^\Pi)\big|\mathcal{F}_{\tau_k^\Pi}\Big\}\le e^{-\alpha\tau_{k-1}^\Pi}N_{k-1}^\Pi\cdot S(\tau_{k-1}^\Pi).$$

Hence, for any $M>i$

$$\begin{aligned}\mathbb{E}^{\,y}\sum_{k=i}^{M}e^{-\alpha\tau_k^\Pi}C_k^\Pi&\le\mathbb{E}^{\,y}\Big\{\sum_{k=1}^{M-1}e^{-\alpha\tau_k^\Pi}C_k^\Pi+\mathbb{E}^{\,y}\Big\{e^{-\alpha\tau_M^\Pi}\Big(C_M^\Pi+N_M^\Pi\cdot S(\tau_M^\Pi)\Big)\big|\mathcal{F}_{\tau_{M-1}^\Pi}\Big\}\Big\}\\&\le\mathbb{E}^{\,y}\Big\{\sum_{k=1}^{M-1}e^{-\alpha\tau_k^\Pi}C_k^\Pi+e^{-\alpha\tau_{M-1}^\Pi}N_{M-1}^\Pi\cdot S(\tau_{M-1}^\Pi)\Big\}\\&\le e^{-\alpha\tau_i^\Pi}N_i^\Pi\cdot S(\tau_i^\Pi).\end{aligned}$$

By monotone convergence theorem

$$\mathbb{E}^{\,y}\sum_{k=i}^{\infty}e^{-\alpha\tau_k^\Pi}C_k^\Pi\le e^{-\alpha\tau_i^\Pi}N_i^\Pi\cdot S(\tau_i^\Pi),$$

which, by Corollary 6.16, is bounded by

$$\eta\cdot s\,\Big(\mathbb{E}^{\,y}\Big\{e^{8\tau_i^\Pi(\hat\beta-2\alpha)}\Big\}\Big)^{1/4}.$$

Lemma 4.1 implies that $\mathbb{E}^{\,y}\big\{e^{8\tau_i^\Pi(\hat\beta-2\alpha)}\big\}$ tends to zero uniformly for $y\in D$ and $\Pi\in\mathcal{A}(y)$.

The fact that v solves the Bellman equation $v=\mathcal{G}v$ follows by standard arguments. The optimality of the strategy constructed in Corollary 4.5 results from (30). ■

The extension of the class of tractable reward functions resulted in a weaker bound for α, the discount factor. In Corollary 6.13 we imposed only

$$\alpha>\tilde\beta=\sup_{(x,i)\in\mathbb{R}^m\times\{1,\dots,d\}}\mu^i(x).$$

In Theorem 6.18, by contrast, we required

$$\alpha>\hat\beta/2=\sup_{i=1,\dots,d}\big(\frac{1}{2}\sup_{x\in\mathbb{R}^m}\|\sigma^i(x)\|^2+\frac{1}{2}\sup_{x\in\mathbb{R}^m}\mu^i(x)\big),$$

which is still better than in Subsection 6a. Moreover, in comparison to results obtained in the previous subsection the specific form of the price process allowed to cover unbounded functions G.

For completeness of the reasoning we shall drive condition NFL2 for unbounded G.

LEMMA 6.20. Under AF and AG, for $\alpha>\hat\beta$ condition NFL2 is satisfied with $\kappa=2$.

Proof. The part of the condition NFL2 concerning F can be proved in the same way as in subsection 6a, but with the use of Corollary 6.16 instead of Corollary 6.2. Here we only have to derive for a compact set $D \subseteq E$

$$\sup_{y\in D}\sup_{\Pi\in\mathcal{A}(y)} \mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^\Pi}\Big(G(C_k^\Pi)\Big)^2 \le \infty.$$

Fix $y=(\eta,s,x)\in E$ and $\Pi\in\mathcal{A}(y)$. We shall prove that

$$\mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^\Pi}\big(G(C_k^\Pi)\big)^2 \le \frac{A^2}{-\alpha} + 2AB(\eta\cdot s) + B^2(\eta\cdot s)^2.$$

By AG

$$\mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^\Pi}\big(G(C_k^\Pi)\big)^2 = \mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^\Pi}\big(A^2 + 2ABC_k^\Pi + B^2(C_k^\Pi)^2\big).$$

By the proof Lemma 6.19 we have the estimate of the first two terms:

$$\mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^\Pi}\big(A^2 + 2ABC_k^\Pi\big) \le \frac{A^2}{-\alpha} + 2AB(\eta\cdot s).$$

We only need to prove that

$$\mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^\Pi} B^2(C_k^\Pi)^2 \le B^2(\eta\cdot s)^2.$$

By Corollary 6.17 we have

$$\mathbb{E}^{\,y}\Big\{e^{-\alpha\tau_k^\Pi}\big(N_{k-1}\cdot S(\tau_k^\Pi)\big)^2\big|\mathcal{F}_{\tau_{k-1}^\Pi}\Big\} \le e^{-\alpha\tau_{k-1}^\Pi}\big(N_{k-1}\cdot S(\tau_{k-1}^\Pi)\big)^2.$$

Hence, for any $M\ge 1$

$$\begin{aligned}
\mathbb{E}^{\,y}\sum_{k=1}^{M} e^{-\alpha\tau_k^\Pi}\big(C_k^\Pi\big)^2 &\le \mathbb{E}^{\,y}\Big\{\sum_{k=1}^{M-1} e^{-\alpha\tau_k^\Pi}\big(C_k^\Pi\big)^2 + \mathbb{E}^{\,y}\Big\{e^{-\alpha\tau_M^\Pi}\Big(\big(C_M^\Pi\big)^2 + \big(N_M^\Pi\cdot S(\tau_M^\Pi)\big)^2\Big)\big|\mathcal{F}_{\tau_{M-1}^\Pi}\Big\}\Big\} \\
&\le \mathbb{E}^{\,y}\Big\{\sum_{k=1}^{M-1} e^{-\alpha\tau_k^\Pi}\big(C_k^\Pi\big)^2 + e^{-\alpha\tau_{M-1}^\Pi}\big(N_{M-1}^\Pi\cdot S(\tau_{M-1}^\Pi)\big)^2\Big\} \\
&\le \ldots \le (\eta\cdot s)^2.
\end{aligned}$$

By monotone convergence theorem

$$\mathbb{E}^{\,y}\sum_{k=1}^{\infty} e^{-\alpha\tau_k^\Pi}\big(C_k^\Pi\big)^2 \le (\eta\cdot s)^2.$$

■

6c. Applications

A considerable number of examples was cited in the introduction. Here we shall present other important examples in detail.

Paper [12] encouraged us to construct a functional that measures diversification of the portfolio. Let $w^* \in \mathcal{S}$, $\mathcal{S} = \{w \in [0,1]^d \ : \ w^1 + \cdots + w^d = 1\}$, be the target proportion (understood as a perfect diversification strategy). Consider the functional

$$\mathbb{E}^{(s,x)} \int_0^\infty e^{-\alpha t} F\big(N^\Pi(t), S(t), X(t)\big) dt, \tag{31}$$

where

$$F(\eta, s, x) = \eta \cdot s \sum_{i=1}^d \alpha^i (1 - |w^i - w^{*i}|),$$

or, in general,

$$F(\eta, s, x) = (\eta \cdot s)\ f(w, w^*, x),$$

where $w^i = \frac{\eta^i s^i}{\eta \cdot s}$, $\alpha^i \geq 0$, $i = 1, \ldots, d$ and f is any continuous bounded function. Observe that assumptions UF and AF are satisfied, so theorems from Sections 6a and 6b easily apply.

In the model of Section 6b we consider a problem of variance-conscious portfolio management. The reward functional has the form (31) with

$$F(\eta, s, x) = (\eta \cdot s)\ f\Big(\|\sum_{i=1}^d w^i \sigma^i(x)\|_2\Big)$$

or, more general, with f depending on the economic factors

$$F(\eta, s, x) = (\eta \cdot s)\ f\Big(\|\sum_{i=1}^d w^i \sigma^i(x)\|_2, x\Big),$$

where w is the proportion as above, $\sigma(x)$ is a volatility matrix of the price process (see (21)), $\| \cdot \|_2$ is an Euclidean norm in $\mathbb{R}^p$ and f is a bounded continuous function, e.g.

$$f(p) = (\Delta - p)^\gamma + \delta, \quad p \in \mathbb{R},$$

where $\gamma > 0$, $\delta \geq 0$ and

$$\Delta = \sup_{x \in \mathbb{R}^m} \sup_{i=1,\ldots,m} |\sigma^i(x)|.$$

Here, same as above, assumptions UF and AF are satisfied.

References

1. Bensoussan A, Lions JL (1982) Contrôle Impulsionnel Inéquations Quasi-Variationnelles. Dunod, Paris

2. Bielecki TR, Pliska SR, Sherris M (2004) Risk sensitive asset allocation. J. Econ. Dyn. Control 24: 1145-1177

3. Buckley IRC, Korn R (1998) Optimal index tracking under transaction costs and impulse control. Int. J. Theor. Appl. Finance 1.3: 315-330
4. Constantinides GM, Richard SF (1978) Existence of optimal simple pilicies for discounted-cost inventory and cash management in continuous time. Oper. Res. 26.4: 620-636
5. Dynkin EB (1965) Markov processes. Springer
6. Eastham JF, Hastings KJ (1988) Optimal impulse control of portfolios. Math Oper Res 13.4: 588 - 605
7. Hernández-Lerma O (1989) Adaptive Markov Control Processes. Springer
8. Korn R (1998) Portfolio optimization with strictly positive transaction cost and impulse control. Finance Stochast 2: 85 - 114
9. Mackevicius V (1973) Passing to the limit in the optimal stopping problems of Markov processes. ibidem 13.1:115 - 128
10. Mackevicius V (1974) Convergence of cost functions in optimal stopping problems of Markov processes. Liet. Mat. Rink. 14:113-128
11. Morton AJ, Pliska SR (1995) Optimal portfolio management with fixed transaction costs. Math. Finance 5:337 - 356
12. Pliska SR, Suzuki K (2004) Optimal tracking for asset allocation with fixed and proportional transaction costs. Quantitative Finance 4.2: 233-243
13. Stettner Ł (1989) On some stopping and impulsive control problems with a general discount rate criteria. Prob Math Stat 10.2: 223 - 245
14. Vollert A (2003) A Stochastic Control Framework for Real Options in Strategic Valuation. Birkhäuser
15. Yao DD, Zhang S, Zhou XY Tracking a Financial Benchmark Using a Few Assets. to appear in Oper. Res.
16. Zabczyk J (1983) Stopping problems in stochastic control. Proceedings of the International Congress of Mathematicians, Warsaw: 1425-1437