Homework

- due on January 15 no later than 11:59 a.m.;
- turn in directly to me or to my mailbox on the 3rd floor (4th including the ground floor);
- you can also turn in good quality scans by e-mail.

Problem 1 *is offered* **only** *to students who got less than* 35 *points from the mid-term exam. The rest of the problems are* **intentionally** *vaguely stated. You are to investigate and try to find suitable answers. Explaining why your answer is suitable is a part of the problem. If not certain about the direction of your solution feel free to ask me questions via e-mail or during the office hours (this is also intended). Partial solutions will yield some points. Particularly good solutions will yield extra points (for example removing the assumption of compact support for* μ *in* **Problem 3***).*

Problem 1. (a) Let μ be a measure on $\mathbb{X}$ and $f_n, f : \mathbb{X} \to \mathbb{R}$ be μ-summable such that $f_n \xrightarrow{\mu} f$. Prove that $\sqrt{|f_n|} \xrightarrow{\mu} \sqrt{|f|}$.

(b) Let μ be **any** measure on $\mathbb{R}^n$ and $f \in C_b(\mathbb{R}^n)$ (continuous and bounded). Prove that **for all** $x \in \mathbb{R}^n$ we have

$$\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, \mathrm{d}\mu = f(x).$$

Problem 2. (a) Find a Radon measure μ and a μ-summable function f on $\mathbb{X} = \mathbb{R}$, such that the identity

$$\lim_{y\to 0} \int_{\mathbb{X}} (f(x+y) - f(x)) \, \mathrm{d}_x\mu = 0 \tag{1}$$

fails.

(b) With $\mathbb{X} = \mathbb{R}^d$, find reasonably weak assumptions on μ ensuring that (1) holds for all μ-summable functions f. Strict optimality of the assumptions would be nice, but it is not necessarily what you should aim for.

Problem 3. Construct a framework to approximate any finite, compactly supported Radon measure on $\mathbb{R}^d$ with linear combinations of Dirac deltas. More precisely: let μ be any finite, compactly supported Radon measure on $\mathbb{R}^d$. When is it true that you can find a sequence of measures μ_n of the form

$$\mu_n = \sum_{i=1}^{N_n} m_i^n \delta_{x_i^n}, \quad \{x_i^n\}_{i=1}^{N_n} \subset \mathbb{R}^d,$$

such that $\mu_n \to \mu$ in some sense? Deciding on the proper topology is a part of the problem.

Problem 4. Let μ be a finite Radon measure on $\mathbb{R}^d$. Characterize $(\mu \otimes \mu)(\mathcal{D})$, where

$$\mathcal{D} \; = \; \{(x,y) \in \; \mathbb{R}^{2d} : x = y\}.$$