Problem Set 8

Problem 1. Let μ be a signed measure on $\mathbb{X}$. Prove that

$$|\mu| = \sup_{\phi \in C_b(\mathbb{X})} \phi \, \mathrm{d}\mu.$$

Problem 2. Let $\{\mu_n\}$ be a tight sequence of finite signed measures on $\mathbb{R}^d$. Prove that $\mu_n \to \mu$ narrowly for some finite signed measure μ iff $d(\mu_n, \mu) \to 0$ and $\{\mu_n\}$ is bounded in $|\cdot|$.

Problem 3. Let A be a λ^d-measurable subset of $\mathbb{R}^d$. Describe the Hausdorff dimension of A ($\dim_{\mathrm{H}}(A)$) if

(a) A is at most countable,

(b) $0 < \lambda^d(A) < \infty$,

(c) $\lambda^d(A) = \infty$.

Problem 4. Prove that if there exists a probability measure μ on $\mathbb{X} \supset A$ and positive constants C, d, such that for all $\epsilon > 0$ and all $x \in A$, we have

$$\mu(B(x, \epsilon)) \leq C\epsilon^d$$

then $\dim_{\mathrm{H}}(A) \geq d$.

Problem 5. Using the previous problem show that if $f : [0, 1] \to [0, 1]$ is a nondecreasing α-Hölder continuous function, then for any $A \subset [0, 1]$ we have $\dim_{\mathrm{H}} f(A) \leq \frac{1}{\alpha} \dim_{\mathrm{H}} A$.

Problem 6. Prove that the Hausdorff dimension of the Cantor set is $\frac{\ln 2}{\ln 3}$.

Problem 7. Consider the Koch curve (and Koch snowflake) defined at
`https://en.wikipedia.org/wiki/Koch_snowflake`.
We shall refer to the boundary of the Koch snowflake as the Koch curve.

(a) Prove that the Koch curve is well-defined.

(b) Calculate the perimeter of the snowflake (the good, old fashioned length of the curve).

(c) Calculate the area (λ^2) of the snowflake.

Problem 8. Find the Hausdorff dimension of the Koch curve and of the Koch snowflake.