Problem Set 7

This set is dedicated to the rectification and clarification of solutions to the previous set. Mostly.

Definition 1. *We say that $\nu : 2^{\mathbb{R}^d} \to [-\infty, +\infty]$ is a* <u>*signed measure*</u> *(on $\mathbb{R}^d$) if*

1. *there exists a Radon measure μ on $\mathbb{R}^d$ and a locally μ-summable function $f : \mathbb{R}^d \to [-\infty, +\infty]$, such that*

$$\nu(K) = \int_K f \, \mathrm{d}\mu$$

for all compact sets $K \subset \mathbb{R}^d$,

2. *for all $A \subset \mathbb{R}^d$, there exists a Borel set $B \supset A$ such that $\nu(A) = \nu(B)$.*

Definition 2. *Let ν be a signed measure. We denote*

$$|\nu| := \sup_B |\nu(B)|,$$

where the supremum is taken over all Borel sets B.

Problem 1. Prove that if $|\nu| = 0$ then $\nu = 0$, *i.e.* for all $A \subset \mathbb{R}^d$ we have $\nu(A) = 0$.

Problem 2. Let $\{\mu_n\}$ be a sequence of Radon measures. Suppose that for all Borel sets B, the limit $\lim_{n\to\infty} \mu_n(B)$ exists. Prove that the limit $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ exists for all $A \subset \mathbb{R}^d$.

Problem 3. Let $\{\mu_n\}$ and μ be as in the previous problem. Suppose that $|\mu_n - \mu| \to 0$. Prove that μ is a Radon measure.

Problem 4. Suppose that $\{\mu_n\}$ is a Cauchy sequence of Radon measures (in $|\cdot|$). Then, we know that it is convergent in $|\cdot|$ to a Radon measure μ, defined in Problem 2 (why?). Prove that

$$\tau := \sum_{n=1}^{\infty} \frac{\mu_n}{2^n}$$

is well defined as a limit in $|\cdot|$. Prove that it is a Radon measure, such that $\mu_n << \mu$ for all n.

Problem 5. Let $\nu << \mu$ be two Radon measures and let $f \in L^1(\nu)$. Is it necessarily true that $f D_\mu \nu \in L^1(\mu)$?

Problem 6. Let $\{\nu_n\}$ be a Cauchy sequence of signed measures (in $|\cdot|$ norm). Assume further that each of ν_n is represented (according to Definition 1) as

$$\nu_n(K) = \int_K f_n \, \mathrm{d}\mu,$$

where μ is a certain Radon measure. Prove that $\{f_n\}$ is a Cauchy sequence in $L^1_K(\mu)$ for each compact set K.