## Problem Set 6

**Problem 1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be an upper semicontinuous, *i.e.* $\limsup_{x_n \to x} f(x_n) \leq f(x)$. Prove that

(a) $\{x : \ f(x) < y\}$ is open for all $y \in \mathbb{R}$,

(b) $f$ is measurable with respect to all Radon measures.

**Problem 2.** Let $\nu << \mu$ be two Radon measures. Prove that $\mu$-measurable sets are $\nu$-measurable. You can also think about the inverse implication.

**Problem 3.** Let $\lambda^1$ be Lebesgue measure on $\mathbb{R}$. Calculate $D_{\lambda^1}\nu$ directly from definition:

(a) $\nu = \lambda^1 \llcorner [0, 1]$,

(b) $\nu : \nu(A) = \int_A f \, \mathrm{d}\lambda^1$ for all Borel sets $A$ and a locally summable function $f \geq 0$.

(c) $\nu = \delta_0$, where $\delta_0$ is a Dirac measure supported in 0.

**Problem 4.** Let $\mu$ and $\nu$ be Radon measures and let $f \geq 0$ be a $\mu$-summable function. Assume further that

$$\nu(A) = \int_A f \, \mathrm{d}\mu \qquad \text{for all Borel } A.$$

Find $D_\mu \nu$.

**Problem 5.** Let $\{f_n\}$ be an increasing sequence of nonnegative $\mu$-measurable functions convergent in $\mu$ to $f$. Is it necessarily true that $f \geq f_n$ for all $n$?

**Problem 6.** Below, $f$ is always a function from $[0, 1]$ to $\mathbb{R}$. Measure is always $\lambda^1$.

(a) Provide an example of a continuous function $f$ that is not absolutely continuous.

(b) Provide an example of an a.e. continuous function $f$, such that $f'$ exists a.e. and $f' = 0$ a.e. but also $f(0) = 0$ and $f(1) = 1$.

(c) Provide an example of a function $f$ that is not continuous at any point but is a.e. equal to a smooth function.

(d) Provide an example of an everywhere continuous function $f$ that is nondecreasing and differentiable a.e. with $f' = 0$ a.e. but also $f(0) = 0$ and $f(1) = 1$.

(e) **(For those who know some PDEs)** Understand why weak derivative is great.

**Problem 7.** Provide an example of a non-atomic (nonzero) measure $\mu$ mutually singular with $\lambda^1$. Non-atomic means that $\mu(\{x\}) = 0$ for all $x$.

**VERTE**

**In what follows it is enough to assume that $\mathbb{R}^d$ is just $\mathbb{R}$.**

**Definition 1.** *We say that $\nu$ is a signed measure (on $\mathbb{R}^d$) if there exists a Radon measure $\mu$ on $\mathbb{R}^d$ and a locally $\mu$-summable function $f : \mathbb{R}^d \to [-\infty, +\infty]$, such that*

$$\nu(K) = \int_K f \, \mathrm{d}\mu$$

*for all compact sets $K \subset \mathbb{R}^d$.*

**Definition 2.** *Let $\nu$ be a signed measure. We denote*

$$|\nu| := \sup_A |\nu(A)|$$

*where the supremum is taken over all $\nu$-measurable sets $A$.*

**Problem 8.** Let $\mathcal{M}$ be the set of all signed measures. Prove that

(a) $\mathcal{M}$ is a linear space over $\mathbb{R}^d$ (don't follow through every detail),

(b) $|\cdot|$ is a norm on $\mathcal{M}$,

(c) $(\mathcal{M}, |\cdot|)$ is a Banach space. **This one is probably more difficult.**

**Definition 3.** *Let $\mu$ and $\nu$ be two probability measures on $\mathbb{R}^d$. We denote*

$$d(\mu, \nu) := \sup_g \left\{ \left| \int_{\mathbb{R}^d} g \, \mathrm{d}\mu - \int_{\mathbb{R}^d} g \, \mathrm{d}\nu \right| \right\},$$

*where the supremum is taken over all smooth functions $g$, $\|g\|_\infty \leq 1$ with the Lipschitz constant $\mathrm{Lip}(g) \leq 1$.*

**Problem 9** (Easy)**.** Prove that $d$ is a metric.

**Problem 10.** Prove that for any two probability measures $\mu$ and $\nu$ we have $d(\mu, \nu) \leq C|\mu - \nu|$.

**Problem 11** (Easy)**.** For $x$ and $y$ in $\mathbb{R}^d$ calculate $|\delta_x - \delta_y|$ and $d(\delta_x, \delta_y)$. What happens if $y \to x$?