Problem Set 5

Problem 1. Calculate the λ^n Lebesgue measure of the sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} \subset \mathbb{R}^n$.

Problem 2. Let $\mathcal{F}$ be a family of real, μ-integrable functions on $\mathbb{X}$. Prove that $\mathcal{F}$ is uniformly summable provided that:

(a) there exists a summable function g such that $|f| \leq g$ μ-a.e. for all $f \in \mathcal{F}$;

(b) $\sup_{f\in\mathcal{F}} \int_{\mathbb{X}} |f|^p \, \mathrm{d}\mu < \infty$ with $p > 1$, provided that $\mu(\mathbb{X}) < \infty$.

Problem 3. Let μ be a σ-finite measure. Let $\{f_n\}_{n=1}^{\infty}, f$ be μ-integrable functions such that $f_n \to f$ μ-a.e. Assume further that there exists a sequence of μ-summable functions $\{g_n\}$ convergent to a μ-summable function g in $L^1(\mu)$ and in measure μ, such that $|f_n| \leq |g_n|$ for all n. Prove that $f_n \to f$ in $L^1(\mu)$.

Problem 4. Provide an example of a sequence of λ^1-summable functions $\{f_n\}$ such that

(a) $f_n \to f$ in measure, but not μ-a.e.;

(b) $\{f_n\}$ is uniformly summable but it is neither monotone nor is it dominated by a λ^1-summable function.

Problem 5. Consider the Fatou lemma and the monotone convergence theorem with the "μ-a.e." convergence replaced with the "in measure" convergence.