Problem set 3

If not stated otherwise μ is a measure defined on $\mathbb{X}$.

Problem 1 Let $f : \mathbb{X} \to \mathbb{R}$ be μ-measurable and bounded. Prove that there exists a sequence of simple functions convergent to f uniformly on $\mathbb{X}$. *Note that we know (almost) nothing except the definition.*

Problem 2 Let $f : \mathbb{X} \to \mathbb{R}$ be μ-measurable and nonnegative. Prove that there exists a sequence of simple functions convergent to f μ-a.e. Is the assumption of nonnegativity of f essential?

Problem 3 Prove that $f(x) = \frac{1}{x}$ is λ^1-integrable in $(0, +\infty)$. Find $\int_0^\infty f \, \mathrm{d}\lambda^1$. *Note that we know (almost) nothing except the definition.*

Problem 4 Prove that in the monotone convergence theorem nonnegativity of function f_1 (which implies nonnegativity of the entire sequence $\{f_n\}_n$) can be replaced by summability of f_1.

Definition 1 *We say that $x \in \mathbb{X}$ is an atom if $\mu(\{x\}) > 0$.*

Problem 5 Let μ be a Borel measure. Prove that if μ has an uncountable set of atoms then $\mu(\mathbb{X}) = +\infty$.

Problem 6 Let μ be a probability measure (Radon measure with $\mu(\mathbb{X}) = 1$). Let $\{f_n\}_n$ be a sequence of μ-measurable functions convergent μ-a.e. to a μ-summable function f. Assume further that

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, \mathrm{d}\mu = \int_{\mathbb{X}} f \, \mathrm{d}\mu.$$

Prove that for any $\epsilon > 0$, there exists a μ-measurable set E and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\Big| \int_{\mathbb{X}\setminus E} f_n \, \mathrm{d}\mu \Big| \leq \epsilon \quad \text{and} \quad |f_n(x)| \leq |f(x)| + 1 \quad \text{for} \quad x \in E.$$