Problem set 2

If not stated otherwise μ is a measure defined on $\mathbb{X}$.

Problem 1 Let $\mathcal{U}$ be a family of open sets in $\mathbb{R}^n$. Prove that there exists an at most countable family $\mathcal{S} \subset \mathcal{U}$ with the same union as $\mathcal{U}$.

Problem 2 Let U be a nonempty open subset of $\mathbb{R}^n$. Prove that U is a union of at most countable collection of cubes with edges parallel to the coordinate axes.

Problem 3 Prove that $\mathcal{B}(\mathbb{R}) = \sigma(\{\text{ open intervals in } \mathbb{R}\})$. Here and hereinafter $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel sets in $\mathbb{R}$.

Problem 4 Provide an example of a metric space $\mathbb{X}$ for which the assertion of Problem 1 fails.

Problem 5 (Back to Problem 12 from set 1) With the assumptions of Problem 12 from set 1 suppose that we know that $\mu(B) = \nu(B)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$. Prove that $\mu(A) = \nu(A)$ for all $A \subset \mathbb{R}^n$.

Problem 6 Let $f : \mathbb{X} \to \mathbb{Y}$. where $\mathbb{Y}$ is a topological space, be a measurable function. Prove that the preimage $f^{-1}(B)$ of each Borel set B is μ-measurable.

Problem 7 Let $f : \mathbb{X} \to \mathbb{R}$ be a function such that for all $a \in \mathbb{R}$. $f^{-1}((-\infty, a])$ is μ-measurable. Prove that f is a measurable function.

Problem 8 For $A \subset \mathbb{R}$ let

$$\lambda^1(A) = \inf\left\{\sum_{i=1}^{\infty} \mathrm{diam} C_i : \ A \subset \bigcup_{i=1}^{\infty} C_i, \ C_i \subset \mathbb{R}\right\}.$$

(a) Prove that λ^1 is a measure.

(b) Find $\lambda^1([0,1])$.

Definition 1 *We say that a subset A of a topological space is of <u>first category</u> if it is a union of an at most countable collection of nowhere dense sets.*

Problem 9 Let μ be a finite Borel regular measure defined on a closed interval $I \subset \mathbb{R}$. Prove that I can be decomposed as a union of a first category set and a μ-negligible set.

Problem 10 Let $A \subset \mathbb{R}$ be bounded and let μ be a Radon measure on $\mathbb{R}$. Assume further that f is a μ-measurable function finite μ-a.e. on A. Prove that for all $\epsilon > 0$ there exists a measurable set $B \subset A$ with $\mu(A \setminus B) < \epsilon$ such that f is bounded on B. Is the assertion true for a wider class of measures? Maybe Borel regular measures? Or Borel measures?

Problem 11 Prove that the assumption that $\mu(A) < +\infty$ in Egoroff's theorem is essential.

Problem 12 Let A be a μ-measurable subset of $\mathbb{R}^n$ and let $\{f_n\}$ be a sequence of μ-measurable functions converging to the function f μ-a.e. on A. Prove that there exists $B \subset A$, such that $\mu(A \setminus B) = 0$ and B can be decomposed into an at most countable union of sets B_i with $f_n \rightrightarrows f$ on each B_i.

Problem 13 Provide an example of a sequence $\{f_n\}$ of λ^1-measurable functions on $[0,1]$, convergent to the function f a.e on $[0,1]$, with the property that for all $A \subset [0,1]$, $\lambda^1(A) = 1$ the convergence **is not** uniform.

Problem 14 Provide an example of a λ^n-measurable set $A \subset \mathbb{R}^n$ such that for each ball $B \subset \mathbb{R}^n$ we have

$$\lambda^n(A \cap B) > 0 \qquad \text{and} \qquad \lambda^n((\mathbb{R}^n \setminus A) \cap B) > 0.$$

Problem 15 Provide an example showcasing that in Lusin's theorem we cannot assume that K is a set of full measure (i.e. we cannot take $\epsilon = 0$).

Problem 16 (A more difficult version of the previous problem) Provide an example of a measurable function f (with all assumptions of Lusin's theorem satisfied) which is not continuous at any point of any set of the form $\mathbb{R}^n \setminus E$, where E is μ-negligible.

Hint: use the set from Problem 14.