Problem set 1

If not stated otherwise μ is a measure defined on $\mathbb{X}$.

Definition 1 *A family Λ of subsets of $\mathbb{X}$ is a* <u>*λ-system*</u> *if it satisfies the following conditions:*

1. $\mathbb{X} \in \Lambda$,
2. *For all* $A \subset B \in \Lambda$ *we have* $B \setminus A \in \Lambda$,
3. *For all* $\mathcal{A} = \{A_1, A_2, ...\} \subset \Lambda$ *with* $A_i \subset A_{i+1}$ *for all* $i = 1, 2, ...$, *we have* $\bigcup \mathcal{A} \in \Lambda$.

Definition 2 *A family Π of subsets of $\mathbb{X}$ is a* <u>*π-system*</u> *if for all $A, B \in \Pi$ we have $A \cap B \in \Pi$.*

Problem 1 (Lemma's lemma) Prove that if a λ-system Λ is a π-system then it is a σ-algebra.

Problem 2 (Sierpiński) Prove that if a λ-system Λ contains a π-system Π then it contains $\sigma(\Pi)$. Here $\sigma(\Pi)$ is the smallest σ-algebra containing Π.

Problem 3 (Basic examples) Which of the following functions $\mu : 2^{\mathbb{R}} \to [0, +\infty]$ are measures? Explain your reasoning.

(a)
$$\mu(A) = \#A, \tag{1}$$
where $\#A$ is the number of elements of A (with the convention that $\#A = +\infty$, whenever A is infinite).

(b)
$$\mu(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases} \tag{2}$$

(c)
$$\mu(A) = \int_A g(x) d\lambda(x),$$
where λ is the Lebesgue measure and $0 \leq g \in L^1(\lambda)$. Here there may be a problem with the definition of the integral on the non-measurable sets. Think about it.

(d) Let N be a finite set and $\mu(A)$ is the number of elements of N that belong to A.

Problem 4 For each measure from Problem 3 characterize μ-measurable sets.

Problem 5 Let B be μ-measurable. Prove that for all $A \subset \mathbb{X}$, B is $\mu \llcorner A$-measurable. Here $\mu \llcorner A$ is μ restricted to A.

Problem 6 Let μ be a Borel regular measure and let A be a Borel subset of $\mathbb{X}$. Prove that $\mu \llcorner A$ is Borel regular.

Problem 7 (Was on the lecture) Prove that any $A \subset \mathbb{X}$ such that $\mu(A) = 0$ is μ-measurable.

Problem 8 (Open question) Is it true that for all $A, B \subset \mathbb{X}$

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)? \tag{3}$$

Propose a **reasonable** assumption on μ so that (3) holds for all $A, B \subset \mathbb{X}$.

Problem 9 (Closed question) Prove that (3) holds with "=" instead of "$\leq$" for all measures μ and for all μ-measurable A and B.

Problem 10 Let $\mu(A) < +\infty$. Prove the following: *If there exists a measurable $A' \subset A$ such that $\mu(A) = \mu(A')$ then A is measurable.* Do the sanity check by comparing this to the definition of a regular measure.

Problem 11 (An important construction) Let $\emptyset \in \mathcal{C} \subset 2^{\mathbb{X}}$ and let $\nu : \mathcal{C} \to [0, +\infty]$ with $\nu(\emptyset) = 0$. Prove that

$$\mu(A) := \inf \left\{ \sum_{i=1}^{\infty} \nu(E_i) : \{E_i\}_{i \in \mathbb{N}} \subset \mathcal{C}, A \subset \bigcup_{i=1}^{\infty} E_i \right\}$$

is a measure.

Problem 12 (Once upon a time on a final exam) Let μ and ν be a couple of finite Radon measures on $\mathbb{R}^n$. We say that a set P is a compact interval iff

$$P = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]; a_i, b_i \in \mathbb{R}, a_i \leq b_i \text{ for } i = 1, 2, ..., n.$$

Suppose that for all compact intervals P we have $\nu(P) = \mu(P)$. Prove that $\mu(A) = \nu(A)$ for all $A \subset \mathbb{R}^n$.

Problem 13 (Difficult?) Provide an example of a function $\mu : 2^{\mathbb{R}^n} \to [0, +\infty]$ such that

1. $\mu(\emptyset) = 0$,
2. μ is finitely sub-additive, *i.e.* for all $A \subset \bigcup_{i=1}^n A_i$ we have

$$\mu(A) \leq \sum_{i=1}^{n} \mu(A_i).$$

Ok. Now provide an example of such a μ that **<u>is not</u>** a measure.

Hint: search in the set $(L^\infty)^* \setminus (C_b)^*$.