University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Jakub Koncki

Motivic Chern classes and stable envelopes

PhD dissertation

Supervisor
dr hab. Andrzej Weber
Institute of Mathematics
University of Warsaw

Author's declaration:

I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Jakub Koncki

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dr hab. Andrzej Weber

Abstract

Cohomological invariants of the Białynicki-Birula decomposition are the main subject of this dissertation. In particular we study and compare two characteristic classes: the motivic Chern class of a Białynicki-Birula cell and Okounkov's stable envelope. Both of these classes live in the torus–equivariant K–theory. The stable envelope is defined axiomatically, while the motivic Chern class can be constructed explicitly in terms of a resolution of singularities.

The stable envelope depends on an additional parameter called slope. We define the twisted motivic Chern class which also depends on this parameter. Our construction is a combination of the Brasselet–Schürmann–Yokura definition of the motivic Chern class with the ideas coming from the theory of multiplier ideals. To prove that this class is independent of the choice of a resolution of singularities we use the weak factorization theorem.

We prove that if the Białynicki-Birula decomposition is regular enough, then the motivic Chern class coincides (up to normalization) with the stable envelope for a special value of a slope. Moreover, we prove that the twisted motivic Chern class coincides with the stable envelope for all slopes. Finally, we show that the decomposition of a homogenous variety into the Schubert cells is regular enough. This allows to define the stable envelope for homogenous varieties in terms of a resolution of singularities. Our methods are based on the localization theorems and the Lefschetz–Riemann–Roch theorem.

Keywords: Motivic Chern class, Stable envelope, Białynicki-Birula decomposition, Torus action, Equivariant K-theory, Localization theorem, Schubert varieties.

AMS MSC 2010 classification: 14C17, 14M15, 19L47, 14N15.

Streszczenie

Tematem niniejszej rozprawy doktorskiej są kohomologiczne niezmienniki rozkładu Białynickiego–Biruli. W szczególności badamy dwie klasy charakterystyczne: motywiczną klasę Cherna komórki Białynickiego–Biruli i stabilną otoczkę Okounkov'a. Obie te klasy są elementami ekwiwariantnej K–teorii. Stabilna otoczka jest zdefiniowana aksjomatycznie, podczas gdy motywiczna klasa Cherna może być zadana w sposób jawny za pomocą rozwiązania osobliwości.

Stabilna otoczka zależy od dodatkowego parametru nazywanego "slope". W pracy definiujemy skręconą motywiczną klasę Cherna która również zależy od tego parametru. Jej konstrukcja jest połączeniem definicji motywicznej klasy Cherna (Brasselet-Schürmann-Yokura) i idei inspirowanych teorią ideałów mnożników. Aby dowieść, że skręcona klasa nie zależy od rozwiązania osobliwości używamy słabego twierdzenia o faktoryzacji.

W rozprawie dowodzimy, że gdy rozkład Białynickiego–Biruli jest dostatecznie regularny to motywiczna klasa Cherna jest równa stabilnej otoczce dla specjalnej wartości parametru "slope". Ponadto wykazujemy, że skręcona motywiczna klasa Cherna pokrywa się ze stabilną otoczką dla dowolnej wartości parametru "slope". Na koniec dowodzimy, że rozkład rozmaitości jednorodnych na komórki Schuberta jest dostatecznie regularny. Umożliwia to zdefiniowanie stabilnej otoczki dla rozmaitości jednorodnych za pomocą rozwiązania osobliwości. Nasze rozumowania są oparte o twierdzenia o lokalizacji i twierdzenie Lefschetza–Riemanna–Rocha.

Słowa kluczowe: Motywiczna klasa Cherna, Stabilna otoczka, Rozkład Białynickiego-Biruli, Działanie torusa, Ekwiwariantna K-teoria, Twierdzenie o lokalizacji, Rozmaitości Schuberta.

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Contents

Introduction

Algebraic torus action on a complex algebraic variety induces many interesting structures. One of the most important is the Białynicki-Birula decomposition. It is a decomposition of a variety into attracting sets, called Białynicki-Birula cells. The closures of cells may be singular and their properties, in particular their cohomological invariants, were studied by many authors. However not much is known beyond the classical examples, such as Schubert varieties in homogeneous spaces. In this thesis we aim to present and compare two characteristic classes deforming the fundamental class of the closure of the Białynicki-Birula cell. One of these classes is Okounkov's stable envelope and the other is the motivic Chern class. We will consider the case when the torus acts on a smooth, projective variety with finitely many fixed points.

Białynicki-Birula decomposition (BB-decomposition for short) was introduced in a series of papers [**BB73, BB74, BB76**] and further studied by many authors (see e.g. [**CS79, CG83, JS19**] and [**Car02**] for a survey). Suppose that a one dimensional algebraic torus $\mathbb{C}^*$ acts on a variety M. The BB-decomposition assigns to every component of the fixed point set $M^{\mathbb{C}^*}$ a locally closed subvariety called BB-cell. When the variety M is projective, the BB-cells form a decomposition of M. Moreover, if M is smooth and the fixed point set $M^{\mathbb{C}^*}$ is finite then the BB-cells are isomorphic to affine spaces. For M equal to a projective homogenous variety the BB-cells are orbits of a certain Borel subgroup. These varieties are also called Schubert cells. Their cohomological invariants are objects of great interest in enumerative geometry.

Suppose that an algebraic torus A acts on a smooth variety M. We will consider characteristic classes in the equivariant K-theory $K^A(M)$. The localization theorem (see e.g. [**Seg68, Tho92**]) is the main advantage of working in the torus equivariant setting. It states that there is a certain multiplicative system S such that the restriction map induces an isomorphism

$$S^{-1}K^A(M) \simeq S^{-1}K^A(M^A)\,.$$

In many cases (e.g. when the fixed point set M^A is finite) the ring on the left is much simpler than the equivariant K-theory of M.

The motivic Chern class is a product of the program of generalizing characteristic classes of the tangent bundle to the singular case in a functorial way. This program was initiated by a question of Deligne [**Sul71**, Historical Note]. The first answer was given by MacPherson in [**Mac74**] where the Chern-Schwartz-MacPherson class c_{SM} was defined. In the subsequent years many other characteristic classes were constructed. The motivic Chern class mC_y defined in [**BSY10**] generalizes several of these classes, such as Chern-Schwartz-MacPherson c_{SM} class [**Mac74, Alu04, Alu06**], Baum-Fulton-MacPherson Todd class [**BFM75**], Hirzebruch-Todd class [**BSY10**], or L-class [**BSY10**] (see [**SY07**] for a nice survey). It may be thought of as a relative version of the Hirzebruch-Todd genus [**Hir56, BSY10**]. Some of the mentioned characteristic classes have equivariant counterparts, see e.g. [**Ohm06, Web16, AMSS19, FRW21**]. We will consider the torus equivariant version of the motivic Chern class mC_y^A defined in [**AMSS19, FRW21**]. It assigns to every equivariant map of varieties a polynomial over the K-theory of the target. The common feature of many of mentioned characteristic classes are additivity properties with respect to the decomposition of a variety as a union of closed and open subvarieties. Additivity property of the motivic Chern class states that:

$$\mathrm{mC}_y^A(Y \xrightarrow{f} M) = \mathrm{mC}_y^A(Y \xrightarrow{f_{|Z}} M) + \mathrm{mC}_y^A(Y \setminus Z \xrightarrow{f_{|Y\setminus Z}} M) \in K^A(M)[y]\,,$$

for every closed subvariety $Z \subset Y$.

Assume that Y is a possibly singular A-variety and Δ is a $\mathbb{Q}$-Cartier divisor on Y such that the support of Δ contains the singularities of X. In a joint work [**KW22**] we defined the twisted motivic Chern classes $\mathrm{mC}_y^A(Y, \partial Y; \Delta)$ in the equivariant K-theory of Y. Their construction is a combination of the Brasselet-Schürmann-Yokura definition [**BSY10**] with the ideas coming from the theory of multiplier ideals, see [**Laz04**, §9]. The twisted motivic Chern class can also be interpreted as a limit of the elliptic class constructed by Borisov and Libgober for Kawamata log-terminal pairs [**BL03, BL05**]. The definition of our classes is explicit in terms of a resolution of singularities. We apply the weak factorisation theorem [**AKMW02, Wło09**] to show that the resulting class is well defined.

Stable envelopes are characteristic classes defined initially for symplectic resolutions (see [**Bea00, Kal09**]). They are important objects which derive from geometric representation theory and connect it with enumerative geometry (see surveys [**Oko18a, Oko18b**]). In [**MO19**] they were used to determine quantum multiplication on Nakajima quiver varieties. They have rich connections to various areas of mathematics such as: symplectic duality [**RSVZ19a, RSVZ19b, RW20a**], derived categories [**SZZ21**], quantum integrable system [**RTV15, RTV19**], and combinatoric of puzzles [**KZJ21**].

Stable envelopes were defined in three types: cohomological [**MO19**], K-theoretic [**OS16, Oko17**] and elliptic [**AO21**]. Their definition is still evolving, see [**Oko21**] for a recent progress. In this dissertation we focus on the K-theoretic stable envelopes. They depend on an additional parameter, a fractional line bundle called slope. Their definition is axiomatic and only particular examples were studied in detail (e.g. [**Smi20, SZ20**]). We will consider the case when the symplectic manifold is the cotangent bundle of a smooth variety on which the torus acts with finitely many fixed points. In this context stable envelope assigns to every fixed point a class in the equivariant K-theory which deforms the fundamental class of Białynicki-Birula cell. We construct the stable envelope (for an arbitrary slope) via resolution of singularities of the closure of BB-cell.

One of the important notions in the theory of K-theoretic stable envelopes is the wall R-matrix defined in [**OS16**]. It measures the dependence of the stable envelope on the slope. It turns out that this dependence is locally constant. There exists a division of the linear space of fractional line bundle into open regions (called alcoves) such that the stable envelope depends only on an alcove containing slope. For any choice of a slope the stable envelopes form a basis of a certain free module (i.e. the localized K-theory). The wall R-matrix between two slopes is a base change matrix between bases corresponding to these slopes. In [**SZZ21**] this matrix was computed for a generalized flag variety.

The idea of connecting characteristic classes with stable envelopes originates from parallelly written papers [**RV18**], [**FR18**] and [**AMSS17**]. There, the Chern-Schwartz-MacPherson classes in the equivariant cohomology were compared with cohomological stable envelopes. We focus on the comparison in the K-theory. Let us state our main results. Let M be a smooth projective A-variety. Suppose that the fixed point set M^A is finite. Choose a general enough one parameter subgroup $\sigma : \mathbb{C}^* \to A$. Let

$$M = \bigsqcup_{\mathbf{e} \in M^A} M_{\mathbf{e}}^+$$

be the corresponding Białynicki-Birula decomposition. The twisted motivic Chern classes applied to the closures of cells, with a suitably chosen divisors $\Delta_{\mathbf{e},s}$, satisfy all but one of the axioms of Okounkov's stable envelopes without any additional assumptions.

Theorem (see Theorem 4.2). *The (rescaled) twisted classes*

$$i_* \, \mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}, \partial\overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s}) \in K^A(M)[y] \subset K^{A\times\mathbb{C}^*}(T^*M)$$

satisfy the normalization axiom and the Newton inclusion property of stable envelopes (for a slope s).

The remaining axiom is of a different nature. Its validity depends on the regularity of BB-decomposition. We present a sufficient condition for this axiom to hold. Our condition is satisfied by homogenous varieties for a reductive linear group.

Theorem (see Corollary 5.16). *Suppose that M is a homogenous variety. Then the (rescaled) twisted classes*

$$i_* \, \mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}, \partial\overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s})$$

are equal to the stable envelopes for T^*M for a slope s.

For a small anti-ample slope we obtain comparison with the motivic Chern class.

Theorem (see Theorem 2.2). *The (rescaled) motivic Chern classes*

$$\mathrm{mC}_y^A(M_{\mathbf{e}}^+ \subset M) \in K^A(M)[y] \subset K^{A\times\mathbb{C}^*}(T^*M)$$

satisfy the normalization axiom and the Newton inclusion property of stable envelopes (for a small anti-ample slope s).

Theorem (see Corollary 5.16). *Suppose that M is a homogenous variety. Then the (rescaled) motivic Chern classes*

$$\mathrm{mC}_y^A(M_{\mathbf{e}}^+ \subset M)$$

are equal to the stable envelopes for T^*M for a small anti-ample slope s.

The above theorems are generalisations of the previous results of [**AMSS19, FRW21**]. The approach of [**AMSS19**] is based on the study of the Hecke algebra action on the equivariant K-theory of flag varieties, whereas our strategy is similar to [**FRW21, Web17**]. We study the limits of Laurent polynomials (of one or many variables) to directly check the stable envelopes axioms.

There is one more family of characteristic classes, called weight functions, closely connected to the stable envelopes for homogenous varieties in type A (see e.g. [**RTV15, RTV19**]). One should also mention recent works in the elliptic theory [**RW20b, KRW20**]. We will not extend exposition of these developments here.

The comparison of the stable envelopes with the motivic Chern classes has several advantages. Motivic Chern classes have rich functorial properties. In [**AMSS19**] such comparison was used to prove conjectures of Bump, Nakasuji and Naruse [**BN11, BN19, NN16**] concerning the geometry of homogenous varieties over p-adic fields.

Our work allows to give an explicit definition of the stable envelope in terms of a resolution of singularities of the closure of BB-cell. In general such resolution is hard to find, yet in a special case of a generalized flag variety we may use the Bott-Samelson resolution (see e.g. [**BK05**, Chapter 2.2] or [**Dem74**]). The inductive construction of Bott-Samelson varieties allows to obtain recursive formulas which compute characteristic class of a cell from the classes of smaller cells. Formulas of this form were studied in e.g. [**Knu03, AM16, AMSS19, MNS20, RW20b**]. In our next work [**Kon**] we prove such a formula for the twisted motivic Chern class and use it to calculate the wall R-matrix for a generalized flag variety. This approach allows to reinterpret results of [**SZZ21**] in a geometric setting and simplify them.

Contents

This PhD dissertation consists of six chapters and an appendix. Chapter 1 collects necessary results and definitions used in the further parts of thesis. Chapters 2–5 form the main part of the dissertation. Chapters 2, 5 and appendix A are based on the author work [**Kon22**]. Chapters 3 and 4 are based on a joint work with Andrzej Weber [**KW22**].

In chapter 1 we recall definitions and properties of the equivariant K-theory, BB-decomposition, the motivic Chern class and the stable envelope. We also present certain limit procedures useful in the next chapters.

In chapter 3 we define the twisted motivic Chern class and prove its basic properties. The main part of this chapter contains the proof of independence of the twisted classes from the choice of a resolution of singularities.

In chapter 2 we prove that the motivic Chern class satisfies all but one axioms of the stable envelope for the trivial slope. In chapter 4 we prove that the twisted motivic Chern class satisfies all but one axioms of the stable envelope for a slope s.

In chapter 5 we focus on the remaining axiom. It is of a different nature. Its validity depends on the regularity of the BB-decomposition and it does not hold automatically. To even begin we have to assume that the BB-decomposition is a stratification satisfying the Whitney condition A. We prove that the stratification of a homogeneous space by the Schubert cells is regular enough.

Chapter 6 contains an example. We calculate the twisted motivic Chern classes for the Lagrangian Grassmanian $LG(2,4)$ and explicitly check that they satisfy the axioms of stable envelopes.

Appendix A contains a comparison of our definition of the stable envelope with the standard one. We also present a rigorous proof of uniqueness of the stable envelopes adapted to the case of isolated fixed points.

Notations and assumptions

All considered varieties are complex and quasiprojective, i.e. a variety is a reduced, finite type, quasiprojective scheme over $\mathbb{C}$ (not necessarily irreducible). We use the following notational conventions:

- Y, Z, W are varieties.
- M, N are smooth varieties.
- $A \simeq (\mathbb{C}^*)^{\operatorname{rk} A}$ is an algebraic torus.
- A-variety is a variety equipped with an action of the torus A.
- $\mathbb{T} = A \times \mathbb{C}^*$ is a product of tori.
- $\mathfrak{h} \in \operatorname{Hom}(\mathbb{T}, \mathbb{C}^*)$ is a character of $\mathbb{T}$ given by the projection on the factor $\mathbb{C}^*$.
- $X = T^*M$ is the cotangent variety. If A acts on M we consider the induced action of $\mathbb{T}$ on X (see example 1.61).
- $\sigma \colon \mathbb{C}^* \to A$ is a one parameter subgroup.
- $\mathbf{e}, \mathbf{e}'$ (and variations) are torus fixed points.
- For a point $\mathbf{e} \in M$ we denote by $T_{\mathbf{e}}M$ the tangent space to M at $\mathbf{e}$, i.e. $T_{\mathbf{e}}M = TM_{|\mathbf{e}}$.
- The symbol $\subset$ denotes not necessarily strict inclusion.
- G is a reductive linear complex Lie group. B is a Borel subgroup of G and P is a parabolic subgroup of G. G/B denotes the generalized flag variety of G, G/P denotes a homogenous variety of G. See e.g. [**Bor91**] for an introduction to algebraic groups.
- Let R be a ring and G a group. Then $R[G]$ denotes the group algebra.

CHAPTER 1

Tools

1.1. Equivariant K-theory

In this section we will recall the basic properties of the equivariant K-theory. Our main reference is [**CG10**, Chapter 5]. Let $A \simeq (\mathbb{C}^*)^r$ be an algebraic torus. Let Y be a quasiprojective A-variety. Categories $Vect^A(Y)$ of A-vector bundles on Y and $Coh^A(Y)$ of A-coherent sheaves have distinguished class of short exact sequences.

Definition 1.1. Let Y be a quasiprojective A-variety. The equivariant K-theory of vector bundles $K^A(Y)$ is defined by

$$K^A(Y) = \mathbb{Q}\{\text{isomorphism classes of } A\text{-vector bundles}\}/\sim$$

where $\sim$ is an equivalence relation induced by $[E_1] = [E_0] + [E_2]$, whenever there exists a short exact sequence

$$0 \to E_0 \to E_1 \to E_2 \to 0\,.$$

There is a ring structure on the group $K^A(Y)$ induced by the tensor product

$$[E_1] \cdot [E_2] = [E_1 \otimes E_2].$$

Definition 1.2. Let Y be a quasiprojective A-variety. The equivariant K-theory of coherent sheaves $G^A(Y)$ is the Grothendieck group of the abelian category $Coh^A(Y)$. Explicitly

$$G^A(Y) = \mathbb{Q}\{\text{isomorphism classes of } A\text{-coherent sheaves}\}/\sim$$

where $\sim$ is an equivalence relation induced by $[\mathcal{F}_1] = [\mathcal{F}_0] + [\mathcal{F}_2]$, whenever there exists a short exact sequence

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0\,.$$

There is a $K^A(Y)$-module structure on the group $G^A(Y)$ induced by the tensor product

$$[E] \cdot [\mathcal{F}] = [E \otimes \mathcal{F}].$$

Example 1.3 ([**CG10**, Paragraph 5.2.1]). For $Y = pt$ we have

$$K^A(pt) \simeq G^A(pt) \simeq \mathbb{Q}[\mathrm{Hom}(A, \mathbb{C}^*)] \simeq \mathbb{Q}[t_1^{\pm}, ..., t_r^{\pm}]\,,$$

where $\mathbb{Q}[\mathrm{Hom}(A, \mathbb{C}^*)]$ is a group algebra.

For a character $\alpha \in \mathrm{Hom}(A, \mathbb{C}^*)$ we denote by $t^\alpha \in K^A(pt)$ the class of corresponding one dimensional representation of A.

Let Y be an A-variety and $\mathcal{L} \to Y$ an A-equivariant line bundle. Suppose that $\mathbf{e} \in Y^A$ is a fixed point. Then the fiber $\mathcal{L}_\mathbf{e}$ is a representation of A. By

$$\mathrm{w}_\mathbf{e}(\mathcal{L}) \in \mathrm{Hom}(A, \mathbb{C}^*)$$

we denote the weight of this representation. The weight is a locally constant function on Y^A, that is if $\mathbf{e}$ and $\mathbf{e}'$ are two fixed points belonging to the same component of Y^A, then $\mathrm{w}_{\mathbf{e}}(\mathcal{L}) = \mathrm{w}_{\mathbf{e}'}(\mathcal{L})$. Therefore, it makes sense to define $\mathrm{w}_F(\mathcal{L})$ for a component $F \subset Y^A$.

Let D be an A-equivariant Cartier divisor on Y. It induces an element in the equivariant Chow group $A^A_{\dim Y-1}(Y)$. By [**EG98b**, Theorem 1] the first Chern class induces an isomorphism $Pic^A(Y) \simeq A^A_{\dim Y-1}(Y)$. Thus, the line bundle $\mathcal{O}_Y(D)$ is equipped with the natural choice of a linearization. For a smooth fixed point $\mathbf{e} \in D^A$ the weight $\mathrm{w}_{\mathbf{e}}(\mathcal{O}_Y(D))$ is the normal weight to D at $\mathbf{e}$. For a fixed point $\mathbf{e} \in Y^A$ which does not belong to the support of D, we have $\mathrm{w}_{\mathbf{e}}(\mathcal{O}_Y(D)) = 0$.

Let us recall basic functorial properties of the K-theory.

Proposition 1.4.

- *Let $\mathcal{C}$ be a category of quasiprojective A-varieties with A-equivariant maps. The K-theory of vector bundles induces a contravariant functor*
$$K^A(-)\colon \mathcal{C} \to Rings.$$
For a morphism $f : Z \to Y$ and $E \in K^A(Y)$, the pullback map is given by the standard pullback
$$f^*[E] = [f^*E] \in K^A(Z).$$
- *Let $\mathcal{C}$ be a category of quasiprojective A-varieties with A-equivariant proper maps. The K-theory of coherent sheaves induces a covariant functor*
$$G^A(-)\colon \mathcal{C} \to Groups.$$
For a proper morphism $f : Z \to Y$ and $E \in G^A(Z)$, the pushforward map is given by
$$f_*[E] = \sum_{i=0}(-1)^i[R^i f_* E] \in G^A(Y).$$
- *Let $i\colon U \hookrightarrow Y$ be an open embedding of an invariant subvariety. We have a restriction map*
$$i^*\colon G^A(Y) \to G^A(U)\,.$$
- *Let $f\colon Z \to Y$ be a flat morphism. There is a pullback map*
$$f^*\colon G^A(Y) \to G^A(Z)\,.$$

Corollary 1.5. *For any A-variety Y the morphism $Y \to pt$ induces $K^A(pt)$-algebra structure on $K^A(Y)$.*

Theorem 1.6 ([**CG10**, Proposition 5.1.28]). *Let M be a smooth A-variety. There is a canonical isomorphism*
$$K^A(M) \simeq G^A(M)\,.$$

Corollary 1.7. *Let M be a smooth A-varieties and Y a quasiprojective A-variety. Let $f\colon Y \to M$ be a proper equivariant map. We consider the pushforward*
$$f_*\colon K^A(Y) \to K^A(M)$$
given by
$$K^A(Y) \longrightarrow G^A(Y) \xrightarrow{f_*} G^A(M) \simeq K^A(M)\,.$$

Proposition 1.8 (Projection formula [**CG10**, Paragraph 5.3.12]). *Let Z and Y be A-varieties and $f\colon Z \to Y$ a proper A-equivariant map. The map*

$$f_*\colon G^A(Z) \to G^A(Y)$$

is a morphism of $K^A(Y)$ modules, i.e. for $a \in G^A(Z)$ and $b \in K^A(Y)$ we have

$$f_*(a) \cdot b = f_*(a \cdot f^*(b))\,.$$

Let $N \hookrightarrow M$ be a closed embedding of a smooth A-invariant locally closed subvariety. Its normal bundle is

$$\nu(N \subset M) := \operatorname{coker}(TN \to TM_{|N}) \in Vect^A(N)\,.$$

Proposition 1.9 ([**CG10**, Proposition 5.3.15], [**Tho93**]). *Consider a pullback square of A-varieties:*

$$\begin{array}{ccc} T \times_Y Z & \xrightarrow{\tilde{p}} & Z \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow[p]{} & Y \end{array}$$

Suppose that one of the following conditions hold.

(1) *The map p is proper, and π is flat.*
(2) *All varieties are smooth. The maps p and $\tilde{p}$ are closed embeddings. The map π induces an isomorphism of normal bundles*

$$\tilde{\pi}^*\nu(T \subset Y) \simeq \nu(T \times_Y Z \subset Z)\,.$$

Then, we have an equality

$$\pi^* p_* = \tilde{p}_* \tilde{\pi}^* \colon G^A(T) \to G^A(Z)\,.$$

In the next chapters we will need following technical facts:

Proposition 1.10 ([**Har77**, Exercise 8.4 (a)]). *Let M be a smooth A-variety and $E \in Vect^A(M)$ an A-vector bundle. Consider the associated projective bundle*

$$g\colon \mathbb{P}(E) \to M\,.$$

Then

$$g_*[\mathcal{O}_{\mathbb{P}(E)/M}(s)] = 0 \in K^A(M)$$

for $s \in \{-1, -2, ..., -(\operatorname{rk} E - 1)\}$.

Remark 1.11. The above proposition holds in the derived category. We have

$$R^i g_* \mathcal{O}_{\mathbb{P}(E)/M}(s) = 0$$

for $s \in \{-1, -2, ..., -(\operatorname{rk} E - 1)\}$ and arbitrary i.

Proposition 1.12 ([**CG10**, Formula 5.2.4]). *Let F be a quasiprojective variety equipped with the trivial action of the torus A. Then, we have a canonical isomorphism*

$$K^A(F) = K(F) \otimes_{\mathbb{Q}} \mathbb{Q}[\operatorname{Hom}(A, \mathbb{C}^*)]\,,$$

where $K(F)$ is the nonequivariant K-theory of F. Moreover, for a quasiprojective variety F equipped with the trivial action of the torus $\mathbb{T} = A \times \mathbb{C}^*$ we have a canonical isomorphism

$$K^{\mathbb{T}}(F) = K^{\mathbb{C}^*}(F) \otimes_{\mathbb{Q}} \mathbb{Q}[\operatorname{Hom}(A, \mathbb{C}^*)]\,.$$

Consider a variety F equipped with the trivial $\mathbb{C}^*$-action. Every equivariant vector bundle $E \in Vect^{\mathbb{C}^*}(F)$ decomposes as a sum of $\mathbb{C}^*$-eigenspaces

$$E = \bigoplus_{n\in\mathbb{Z}} E_n \,.$$

The sum $E^+ = \oplus_{n>0} E_n$ is called the positive (or attracting) part of E while the the sum $E^- = \oplus_{n<0} E_n$ is called the negative (or repelling) part.

PROPOSITION 1.13. *Let $\sigma\colon \mathbb{C}^* \to A$ be a one parameter subgroup. Suppose that F is an A-variety, such that the induced $\mathbb{C}^*$-action is trivial. Then taking the positive part of a vector bundle induces a map*

$$K^A(F) \to K^A(F)\,.$$

An analogous result holds for the negative part.

PROOF. All A-equivariant maps between A-vector bundles over F preserve eigenspace decomposition. Therefore, any short exact sequence of such bundles decomposes as a sum of short exact sequences of eigenspaces. ☐

PROPOSITION 1.14 ([**CG10**, Proposition 5.2.14]). *Let Y be an A-variety and Z a closed invariant subvariety. Denote by i the inclusion $i\colon Z \hookrightarrow Y$. Then, there is a short exact sequence*

$$G^A(Z) \xrightarrow{i_*} G^A(Y) \longrightarrow G^A(Y \setminus Z) \longrightarrow 0\,.$$

The following notion of support of a K-theory element is necessary to define the stable envelopes.

DEFINITION 1.15. Let M be a smooth A-variety. Consider an element $a \in K^A(M)$ and a closed invariant subvariety $i\colon Z \hookrightarrow M$. We say that $\mathrm{supp}(a) \subset Z$ if and only if a lies in the image of the pushforward map

$$G^A(Z) \xrightarrow{i_*} G^A(M) \simeq K^A(M).$$

Proposition 1.14 implies that $\mathrm{supp}(a) \subset Z$ is equivalent to $a_{|M\setminus Z} = 0$.

REMARK 1.16. Note that for an element $a \in K^A(M)$ the support of a is not a well defined subset of M. We define only the notion $\mathrm{supp}(a) \subset Z$ for a closed subvariety $Z \subset M$.

1.1.1. Characteristic classes. Let Y be a quasiprojective A-variety. We use the lambda operations $\lambda_y\colon K^A(Y) \to K^A(Y)[y]$ defined by:

$$\lambda_y([E]) := \sum_{i=0}^{\mathrm{rk}\, E} [\Lambda^i E] y^i \in K^A(Y)[y]\,.$$

For an element $\alpha \in K^A(Y)$ we define operation $\lambda_\alpha\colon K^A(Y) \to K^A(Y)$ as a composition of λ_y with a map of $K^A(Y)$ which sends y to α, i.e.

$$\lambda_\alpha([E]) := \sum_{i=0}^{\mathrm{rk}\, E} [\Lambda^i E] \alpha^i \in K^A(Y)\,.$$

The operation $\lambda_{-1}\colon K^A(Y) \to K^A(Y)$ applied to the dual bundle is the K-theoretic Euler class. Namely

$$eu(E) = \lambda_{-1}(E^*)\,.$$

For an A-vector bundle $E \in Vect^A(Y)$ we define

$$\det(E) = \Lambda^{\operatorname{rk} E} E \in Vect^A(Y)\,.$$

PROPOSITION 1.17. *Let Y be an A-variety and $E \in Vect^A(Y)$ an equivariant vector bundle. Then*

$$\lambda_{-1}(E^*) = (-1)^{\operatorname{rk} E} \cdot \det(E^*) \cdot \lambda_{-1}(E)\,.$$

Let $N \hookrightarrow M$ be a closed embedding of a smooth A-invariant locally closed subvariety. We denote by $eu(N \subset M)$ the Euler class of the normal bundle i.e.

$$eu(N \subset M) := \lambda_{-1}(\nu^*(N \subset M)) \in K^A(N)\,.$$

For a fixed point $\mathbf{e} \in N^A$ we write $\nu_{\mathbf{e}}(N \subset M)$ for the restriction $\nu(N \subset M)_{|\mathbf{e}}$.

PROPOSITION 1.18 ([**CG10**, Proposition 5.4.10]). *Let M be a smooth A-variety and $i\colon N \hookrightarrow M$ an immersion of a smooth A-invariant closed subvariety. For an element $a \in K^A(N)$ we have*

$$i^* i_*(a) = a \cdot eu(N \subset M).$$

The above proposition implies that $i^*i_*1 = eu(N \subset M)$. It is sometimes possible to give an explicit formula for the pushforward i_*1 in the K-theory of an ambient space.

PROPOSITION 1.19. *Let M be a smooth A-variety and $\bigcup_{i=1}^m D_i$ a SNC divisor (see* [**Kol07**, Definition 3.24]*). Consider the subvariety $D_I = \bigcap_{i=1}^m D_i$. Let $i : D_I \to M$ be an inclusion, then*

$$i_*1 = i_*[\mathcal{O}_{D_I}] = \prod_{i=1}^{m} (1 - \mathcal{O}_M(-D_i)) \in K^A(M)\,.$$

We will prove this result by induction on m. For $m = 1$ the proposition simplifies to the following lemma.

LEMMA 1.20. *Let M be a smooth A-variety and D a codimension one smooth subvariety. Let $i\colon D \to M$ denote an inclusion. For any element $\alpha \in K^A(M)$ we have*

$$\alpha \cdot (1 - \mathcal{O}_M(-D)) = \alpha \cdot i_*(\mathcal{O}_D) \in K^A(M)\,.$$

PROOF. It follows from a short exact sequence

$$0 \to \mathcal{O}_M(-D) \to \mathcal{O}_M \to i_*(\mathcal{O}_D) \to 0\,.$$

□

PROOF OF PROPOSITION 1.19. For $m = 1$ the proposition follows from lemma 1.20. Suppose that the proposition holds for some m. We will prove it for $m + 1$. The inclusion i decomposes:

$$i\colon \bigcap_{i=1}^{m+1} D_i \xrightarrow{\ j\ } D_{m+1} \xrightarrow{\ \iota\ } M\,.$$

The Divisor $\bigcup_{i=1}^m D_i \cap D_{m+1} \subset D_{m+1}$ is SNC, thus the inductive assumption implies that

$$j_*(1) = \prod_{i=1}^{m} \big(1 - \mathcal{O}_{D_{m+1}}(-D_i \cap D_{m+1})\big) \in K^A(D_{m+1})\,.$$

Lemma 1.20 for $\alpha = \prod_{i=1}^m (1 - \mathcal{O}_M(-D_i))$ implies that

$$\begin{aligned} i_*1 = \iota_* j_* 1 = \iota_* \prod_{i=1}^{m} \big(1 - \mathcal{O}_{D_{m+1}}(-D_i \cap D_{m+1})\big) = \iota_*\iota^* \prod_{i=1}^{m} (1 - \mathcal{O}_M(-D_i)) = \\ = \prod_{i=1}^{m} (1 - \mathcal{O}_M(-D_i)) \cdot \iota_*(1) = \prod_{i=1}^{m+1} (1 - \mathcal{O}_M(-D_i))\,. \end{aligned}$$

□

1.1.2. Localization. Localization theorems are the main advantage of working with the torus equivariant cohomology. Roughly speaking, they allow to consider only the restriction of classes to the fixed point set without a loss of essential information.

DEFINITION 1.21. Let $S_A \subset K^A(pt)$ be a multiplicative system of all nonzero elements. The localized K-theory of an A-variety Y denotes the ring $S_A^{-1}K^A(Y)$.

THEOREM 1.22 (Localization theorem [**Tho92**, Theorem 2.1]). *Let Y be a quasiprojective A-variety. The restriction map*

$$S_A^{-1}K^A(Y) \to S_A^{-1}K^A(Y^A)$$

is an isomorphism.

PROPOSITION 1.23 ([**CG10**, Corrollary 5.11.3]). *Let Y be a smooth quasiprojective A-variety. Let F be a component of the fixed point set Y^A. The class $eu(F \subset Y)$ is invertible in the localised K-theory $S_A^{-1}K^A(F)$.*

THEOREM 1.24 (Lefschetz-Riemann-Roch theorem [**CG10**, Theorem 5.11.7], [**Tho92**, Theorem 3.5]). *Let M and N be smooth quasiprojective A-varieties and $f\colon N \to M$ a proper equivariant map. Let F be a component of the fixed point set M^A. Consider an element $a \in K^A(N)$. Then*

$$\frac{(f_*a)_{|F}}{eu(F \subset M)} = \sum_{G \subset f^{-1}(F)\cap N^A} (f_{|G})_* \frac{a_{|G}}{eu(G \subset N)} \in S_A^{-1}K^A(M)\,,$$

where the sum is indexed by the set of components of the fixed point set N^A whose image lies in F.

REMARK 1.25. Analogous theorems hold in other torus equivariant cohomology theories. See [**EG98a**] for Chow groups, [**Seg68**] for the topological K-theory, [**AF21**] for the singular cohomology and [**tD71**] for a generalized equivariant cohomology theory.

1.2. Newton polytopes and the limit map

In this section we recall the definition of a Newton polytope and define the limit map in the K-theory. The notion of a Newton polytope is essential to define the K-theoretical stable envelopes. The limit map is a useful tool allowing to compare various Newton polytopes.

1.2.1. Newton polytopes.

DEFINITION 1.26. Let R be a commutative ring with unit and Λ a lattice of a finite rank. Consider a group algebra $R[\Lambda]$ and a polynomial $f \in R[\Lambda]$. The Newton polytope $\mathcal{N}(f) \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is a convex hull of the lattice points corresponding to the nonzero coefficients of the polynomial f.

EXAMPLE 1.27. Let $R = \mathbb{Z}$ and $\Lambda = \mathbb{Z} \oplus \mathbb{Z}$. The ring $R[\Lambda]$ is the Laurent polynomial ring in two variables x and y. Let

$$f = y^{-2} - xy^2 + 5x^3 - 2x\,,$$

then

$$\mathcal{N}(f) = \text{conv}((0,-2);(1,2);(3,0);(1,0)).$$

This is a triangle with vertices $(0,-2),(1,2)$ and $(3,0)$.

We recall elementary properties of Newton polytopes:

PROPOSITION 1.28. *Let R be a commutative ring with unit. For any Laurent polynomials $f, g \in R[\Lambda]$ we have*

- *(a) $N(f \cdot g) \subset N(f) + N(g)$.*
- *(b) $N(f \cdot g) = N(f) + N(g)$ when the ring R is a domain.*
- *(c) $N(f \cdot g) = N(f) + N(g)$ when the coefficients of the polynomial f corresponding to the vertices of the polytope $N(f)$ are not zero divisors.*
- *(d) $N(f + g) \subset conv(N(f), N(g))$.*
- *(e) Let $\theta \colon R \to R'$ be a homomorphism of rings and $\theta' \colon R[\Lambda] \to R'[\Lambda]$ its extension. Then $N(\theta'(f)) \subset N(f)$.*

To define the stable envelopes we need the notion of a Newton polytope of a K-theory class. We can also define a Newton polytope of a polynomial over K-theory ring. For convenience, we denote both these polytopes by $\mathcal{N}^{\mathrm{A}}$.

DEFINITION 1.29. Let F be a smooth variety equipped with the trivial action of a torus $\mathbb{T} = A \times \mathbb{C}^*$.

- To define the Newton polytope

  $$\mathcal{N}^{\mathrm{A}}(a) \subset \text{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}$$

  of an element $a \in K^{\mathbb{T}}(F)$, we take the lattice $\Lambda = \text{Hom}(A, \mathbb{C}^*)$ and the ring $R = K^{\mathbb{C}^*}(F)$ and use the canonical isomorphism (proposition 1.12)

  $$K^{\mathbb{T}}(F) \simeq K^{\mathbb{C}^*}(F)[\text{Hom}(A, \mathbb{C}^*)]\,.$$

- To define the Newton polytope

  $$\mathcal{N}^{\mathrm{A}}(a) \subset \text{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}$$

of a polynomial $a \in K^A(F)[y]$, we take the lattice $\Lambda = \mathrm{Hom}(A, \mathbb{C}^*)$ and the ring $R = K(F)[y]$ and use the canonical isomorphism (proposition 1.12)

$$K^A(F)[y] \simeq K(F)[y][\mathrm{Hom}(A, \mathbb{C}^*)] .$$

REMARK 1.30. The above notion of Newton polytope generalizes the notion of degree in cohomology (see [**OS16**, Paragraph 2.1.6]).

REMARK 1.31. In definition 1.29 we define a Newton polytope according to the smaller torus A not the whole torus $\mathbb{T}$.

EXAMPLE 1.32. Suppose that $\alpha \in \mathrm{Hom}(A, \mathbb{C}^*)$ is a single character and $a \in K^{\mathbb{T}}(F)$. Then by proposition 1.28 (c) the Newton polytope $\mathcal{N}^{\mathrm{A}}(\alpha \cdot a)$ is equal to the polytope $\mathcal{N}^{\mathrm{A}}(a)$ translated by α. Let $\mathfrak{h} \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ be a character of $\mathbb{T} = A \times \mathbb{C}^*$ given by a projection on the second factor then

$$\mathcal{N}^{\mathrm{A}}(\mathfrak{h} \cdot a) = \mathcal{N}^{\mathrm{A}}(a) .$$

The following proposition justifies use of the same notation for Newton polytopes of a class and of a polynomial.

PROPOSITION 1.33. *Let F be a smooth variety equipped with the trivial action of a torus $\mathbb{T} = A \times \mathbb{C}^*$. Consider the map*

$$\rho \colon K^A(F)[y] \to K^{\mathbb{T}}(F)$$

given by $\rho(y) = -\mathfrak{h}$. Let

$$a = a_0 + a_1 y + ... + a_m y^m \in K^A(F)[y]$$

be a polynomial. Then

$$\mathcal{N}^{\mathrm{A}}(a) = \mathrm{conv}\left(\bigcup_{i=0}^{m} \mathcal{N}^{\mathrm{A}}(a_m)\right) = \mathcal{N}^{\mathrm{A}}(\rho(a)) .$$

1.2.2. Limit map. The idea to study behaviour of various limits of a Laurent polynomial to obtain informations about its Newton polytope is present in the literature e.g. [**Oko17, SZZ20, FRW21**]. We start with a definition of the limit map for a $\mathbb{C}^*$-action.

DEFINITION 1.34. Let F be a smooth variety equipped with the trivial action of the one dimensional torus $\mathbb{C}^*$. We have the canonical isomorphism (see proposition 1.12)

$$K^{\mathbb{C}^*}(F) \simeq K(F) \otimes_{\mathbb{Q}} \mathbb{Q}[\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*)] \simeq K(F)[\mathbf{t}, \mathbf{t}^{-1}] ,$$

where $\mathbf{t}$ denotes the character of the standard $\mathbb{C}^*$-representation. The limit map $\lim_{\mathbf{t}\to 0}$ is defined on a subring $K(F)[\mathbf{t}]$ by killing all positive powers of $\mathbf{t}$

$$\lim_{\mathbf{t}\to 0} \colon K^{\mathbb{C}^*}(F) \dashrightarrow K(F) .$$

Analogously, we may define the limit map for a one parameter subgroup.

DEFINITION 1.35. Let F be a smooth variety equipped with the trivial action of a torus A. Let $\sigma \colon \mathbb{C}^* \to A$ be a one parameter subgroup. We define the limit map $\lim_\sigma$ as a composition

$$\lim_{\sigma} \colon \; K^A(F) \xrightarrow{\sigma^*} K^{\mathbb{C}^*}(F) \overset{\lim_{\mathbf{t}\to 0}}{\dashrightarrow} K(F) \;\; ,$$

where the first map σ^* is the restriction to the one dimensional torus. This map is defined on the preimage of $K(F)[\mathbf{t}] \subset K^{\mathbb{C}^*}(F)$ under the map σ^*.

The above definition may be extended to the ring of polynomials over K-theory, the localised K-theory and polynomials over the localised K-theory. For convenience, we denote all these maps by $\lim_\sigma$.

DEFINITION 1.36. Let F be a smooth variety equipped with the trivial action of a torus A. Let $\sigma\colon \mathbb{C}^* \to A$ be a one parameter subgroup.

- The limit map $\lim_\sigma$ extends to a subring of a localised K-theory $S_A^{-1}K^A(F)$
$$\lim_\sigma\colon S_A^{-1}K^A(F) \dashrightarrow K(F)\,.$$
This map is defined on a subring of elements $f \in S_A^{-1}K^A(F)$ such that f can be expressed as a quotient $\frac{a}{b}$ with
$$a \in (\sigma^*)^{-1}\left(K(F)[\mathbf{t}]\right), \qquad b \in (\sigma^*)^{-1}\left(K(pt)[\mathbf{t}]\right), \qquad b \notin (\sigma^*)^{-1}\left(\mathbf{t}K(pt)[\mathbf{t}]\right)\,.$$
The limit map is defined by applying $\lim_\sigma$ to the numerator and denominator separately.
- The limit maps
$$S_A^{-1}K^A(F)[y] \dashrightarrow K(F)[y], \qquad\qquad K^A(F)[y] \dashrightarrow K(F)[y]\,,$$
are defined as extensions of the limit maps $S_A^{-1}K^A(F) \dashrightarrow K(F)$ and $K^A(F) \dashrightarrow K(F)$.

PROPOSITION 1.37. *The limit map*
$$\lim_\sigma\colon S_A^{-1}K^A(F) \dashrightarrow K(F)\,.$$
is well defined.

PROOF. Consider $f \in S_A^{-1}K^A(F)$ such that $f = \frac{a_1}{b_1} = \frac{a_2}{b_2}$, where a_1, a_2, b_1, b_2 satisfy conditions from the definition of limit map. We need to prove that
$$\frac{\lim_\sigma a_1}{\lim_\sigma b_1} = \frac{\lim_\sigma a_2}{\lim_\sigma b_2}\,.$$
Let
$$\sigma^*(a_1) = \sum_{k=0}^{N} a_{k,1}\mathbf{t}^k; \quad \sigma^*(a_2) = \sum_{k=0}^{N} a_{k,2}\mathbf{t}^k; \quad \sigma^*(b_1) = \sum_{k=0}^{N} b_{k,1}\mathbf{t}^k; \quad \sigma^*(b_2) = \sum_{k=0}^{N} b_{k,2}\mathbf{t}^k\,.$$
Assumptions from the definition of limit map imply that $b_{0,1} \neq 0$ and $b_{0,2} \neq 0$. The limit map is defined by $\lim_\sigma a_i = a_{0,i}$ and $\lim_\sigma b_i = b_{0,i}$ for $i \in \{1,2\}$. We know that
$$\sum_{k=0}^{N} a_{k,1}\mathbf{t}^k \cdot \sum_{k=0}^{N} b_{k,2}\mathbf{t}^k = \sum_{k=0}^{N} a_{k,2}\mathbf{t}^k \cdot \sum_{k=0}^{N} b_{k,1}\mathbf{t}^k\,.$$
Comparison of the coefficient of $\mathbf{t}^0$ proves that $a_{0,1}b_{0,2} = a_{0,2}b_{0,1}\,.$ □

REMARK 1.38. The denominators $\lim_\sigma b_1, \lim_\sigma b_2$ are elements of $K(pt) = \mathbb{Q}$. Therefore, they are invertible in $K(F)$.

Remark 1.39. In [**Kon21**] for a subtorus $\mathbb{C}^* \subset A$ we defined a more general limit map

$$\lim\colon S_A^{-1}K^A(F) \dashrightarrow S_{(A/\mathbb{C}^*)}K^{(A/\mathbb{C}^*)}(F)\,.$$

We used it to prove the equality of the motivic Chern class with the stable envelope for a small antiample slope. We will not use this map here. Instead we apply methods of [**KW22**].

The limit map allows to compare Newton polytopes.

Proposition 1.40 ([**FRW21**, Proposition 3.6]). *Let F be a smooth variety equipped with the trivial action of a torus $\mathbb{T} = A \times \mathbb{C}^*$. Let a and b be two classes in the equivariant K-theory $K^{\mathbb{T}}(F)$. Suppose that b is invertible in the localized K-theory $S_{\mathbb{T}}^{-1}K^{\mathbb{T}}(Y)$. The following conditions are equivalent*

(1) *There is an inclusion of Newton polytopes*
$$\mathcal{N}^{\mathrm{A}}(a) \subset \mathcal{N}^{\mathrm{A}}(b)\,.$$
(2) *For a general enough one parameter subgroup $\sigma\colon \mathbb{C}^* \to A$ the limit*
$$\lim_\sigma \frac{a}{b}$$
exists.

The proposition for $F = pt$ is proven in [**FRW21**, Proposition 3.6]. The generalization given here is straightforward. Analogous result holds for polynomials $a, b \in K^A(F)[y]$.

Let us note several useful properties of the limit map.

Proposition 1.41. *Let F be a smooth variety equipped with the trivial action of a torus A. Let $a \in S_A^{-1}K^A(F)[y]$ and let $\mathcal{L} \in Pic^A(F)$ be an equivariant line bundle. For any one parameter subgroup $\sigma\colon \mathbb{C}^* \to A$ the limit $\lim_\sigma(\mathcal{L}\cdot a)$ exists if and only if the limit $\lim_\sigma(t^{\mathrm{w}_F(\mathcal{L})}\cdot a)$ exists.*

Proposition 1.42. *Let F be a smooth variety equipped with the trivial action of a torus A. Let $x, y \in S_A^{-1}K^A(F)[y]$. The limit map is additive and multiplicative. If the limits $\lim_\sigma x$ and $\lim_\sigma y$ exist then the limits $\lim_\sigma x + y$ and $\lim_\sigma x\cdot y$ also exist. Moreover*

$$\lim_\sigma x + y = \lim_\sigma x + \lim_\sigma y\,, \qquad \lim_\sigma x \cdot y = \lim_\sigma x \cdot \lim_\sigma y\,.$$

Proposition 1.43. *Let F and G be smooth varieties equipped with the trivial action of a torus A. Let $f\colon F \to G$ be a map. Consider $x \in S_A^{-1}K^A(F)[y]$ and $y \in S_A^{-1}K^A(G)[y]$.*

- *The limit map commutes with the pullback f^*, i.e. if the limit $\lim_\sigma y \in K(G)[y]$ exists than the limit $\lim_\sigma f^*y \in K(F)[y]$ also exists. Moreover*
$$\lim_\sigma f^*y = f^*\lim_\sigma y \in K(F)[y]\,.$$
- *Suppose that f is proper. The limit map commutes with the pushforwards f_*, i.e. if the limit $\lim_\sigma x \in K(F)[y]$ exists than the limit $\lim_\sigma f_*x \in K(G)[y]$ also exists. Moreover*
$$\lim_\sigma f_*x = f_*\lim_\sigma x \in K(G)[y]\,.$$

The analogous results hold for the limit maps on $S_A^{-1}K^A(F)$ and $S_A^{-1}K^A(F)[y]$.

1.2.3. Rational exponents. In chapter 4, we will need to consider polynomials with rational coefficients. In this subsection, we extend methods of the previous subsection to this setting.

DEFINITION 1.44. Let R be a commutative ring with unit and Λ a lattice of finite rank. Consider a polynomial $f \in R[\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}]$. The Newton polytope $\mathcal{N}(f) \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is a convex hull of points corresponding to the nonzero coefficients of the polynomial f.

EXAMPLE 1.45. Let $R = \mathbb{Z}$ and $\Lambda = \mathbb{Z}$. For a rational number $q \in \mathbb{Q}$ denote by x^q the corresponding element in $\mathbb{Z}[\mathbb{Q}]$. Let

$$f = x + x^{1/2} - x^{4/3}\,.$$

The polytope $\mathcal{N}(f)$ is an interval $[1/2; 4/3]$.

DEFINITION 1.46. We define the extended K-theory ring of point as

$$\widetilde{K}^A(pt) = \mathbb{Q}[\mathrm{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}]\,.$$

Let F be a variety equipped with the trivial action of a torus A. We define the extended K-theory ring of the variety F as

$$\widetilde{K}^A(F) = K(F) \otimes_{\mathbb{Q}} \widetilde{K}^A(pt) = K(F)[\mathrm{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}]\,.$$

REMARK 1.47. Suppose that $\tilde{A} \to A$ is a covering of tori. Then we have an inclusion

$$K^{\tilde{A}}(F) \subset \widetilde{K}^A(F)\,.$$

Moreover, for any $a \in \widetilde{K}^A(F)$ there exists a torus covering $\tilde{A} \to A$ such that $a \in K^{\tilde{A}}(F)$.

DEFINITION 1.48. Let $f\colon F \to G$ be a map between varieties equipped with the trivial action of a torus A. We define a homomorphism

$$f^* \otimes id_{\widetilde{K}^A(pt)}\colon \widetilde{K}^A(G) \to \widetilde{K}^A(F)\,.$$

Analogously if f is proper and varieties F and G are smooth we define

$$f_* \otimes id_{\widetilde{K}^A(pt)}\colon \widetilde{K}^A(F) \to \widetilde{K}^A(G)\,.$$

Slightly abusing notation we will denote these maps by f^* and f_*, respectively.

DEFINITION 1.49. Let F be a smooth variety equipped with the trivial action of a torus $\mathbb{T} = A \times \mathbb{C}^*$. To define the Newton polytope $\mathcal{N}^{\mathrm{A}}(a) \subset \mathrm{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}$ of an element $a \in \widetilde{K}^{\mathbb{T}}(F)$ we take the lattice $\Lambda = \mathrm{Hom}(A, \mathbb{C}^*)$, the ring $R = \widetilde{K}^{\mathbb{C}^*}(F)$ and use canonical isomorphisms

$$\begin{aligned}\widetilde{K}^{\mathbb{T}}(F) =& K(F) \otimes_{\mathbb{Q}} \widetilde{K}^{\mathbb{T}}(pt)\\ =& K(F) \otimes_{\mathbb{Q}} \left(\widetilde{K}^A(pt) \otimes_{\mathbb{Q}} \widetilde{K}^{\mathbb{C}^*}(pt)\right)\\ =& \left(K(F) \otimes_{\mathbb{Q}} \widetilde{K}^{\mathbb{C}^*}(pt)\right)[\mathrm{Hom}(A, \mathbb{C}^*) \otimes \mathbb{Q}]\,.\end{aligned}$$

Analogously, we define a Newton polytope of a polynomial $a \in \widetilde{K}^A(F)[y]$.

REMARK 1.50. In chapter 4 we will consider only elements of the form $\alpha \cdot a \in \widetilde{K}^{\mathbb{T}}(F)$, where $a \in K^{\mathbb{T}}(F)$ and $\alpha \in \widetilde{K}^{\mathbb{T}}(pt)$ is a single rational character. The Newton polytopes of such class is translation of a lattice polytope (due to proposition 1.28 (c)).

DEFINITION 1.51. Let F be a smooth variety equipped with the trivial A-action. For $A = \mathbb{C}^*$ we define the limit map

$$\widetilde{K}^{\mathbb{C}^*}(F) \dashrightarrow K(F)$$

on a subring

$$K(F)[\mathbb{Q}_{\geq 0}] \subset K(F)[\mathbb{Q}] = \widetilde{K}^{\mathbb{C}^*}(F)\,,$$

by killing all positive powers of $\mathbf{t}$. Moreover, for an arbitrary torus A and a one parameter subgroup $\sigma\colon \mathbb{C}^* \to A$ we define the limit map

$$\lim_{\sigma}\colon\ \widetilde{K}^{A}(F) \xrightarrow{\sigma^*} \widetilde{K}^{\mathbb{C}^*}(F) \overset{\lim_{\mathbf{t}\to 0}}{\dashrightarrow} K(F)\ ,$$

as in definition 1.35.

Exact analogues of propositions 1.43 and 1.42 hold in the rational exponents setting. Moreover, the analogue of proposition 1.40 also holds.

PROPOSITION 1.52 (see [**FRW21**, Proposition 3.6]). *Let F be a smooth variety equipped with the trivial action of a torus $\mathbb{T} = A \times \mathbb{C}^*$. Let a and b be two classes in the equivariant K-theory $\widetilde{K}^{\mathbb{T}}(F)$. Suppose that b is invertible in the localized K-theory $S_{\mathbb{T}}^{-1}\widetilde{K}^{\mathbb{T}}(F)$. The following conditions are equivalent*

(1) There is an inclusion of Newton polytopes

$$\mathcal{N}^{\mathrm{A}}(a) \subset \mathcal{N}^{\mathrm{A}}(b)\,.$$

(2) For a general enough one parameter subgroup $\sigma\colon \mathbb{C}^* \to A$ the limit

$$\lim_{\sigma} \frac{a}{b} \in K(F)$$

exists.

PROOF. We may find a coverings $\tilde{A} \to A$ and $\mathbb{C}^* \to \mathbb{C}^*$ such that $a, b \in K^{\tilde{A}\times\mathbb{C}^*}(F)$ and b is invertible in $S^{-1}_{\tilde{A}\times\mathbb{C}^*}K^{\tilde{A}\times\mathbb{C}^*}(F)$. Then, the proposition follows from 1.40 after identification

$$\mathrm{Hom}(\tilde{A}, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

□

The following proposition reduces computations of limit of a polynomial with rational coefficients to the standard case.

PROPOSITION 1.53. *Consider a variety F with the trivial action of a torus A and an element of the localised K-theory $\alpha \in S_A^{-1}K^A(F)[y]$. Let $w \in \mathrm{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a fractional weight. Let $w_0, ..., w_m \in \mathrm{Hom}(A, \mathbb{C}^*)$ be characters such that*

$$w \in \mathrm{conv}(w_0, ..., w_m) \subset \mathrm{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}\,.$$

Suppose that for a chosen one parameter subgroup σ all of the limits $\lim_\sigma t^{w_i}\alpha$ exist. Then the limit $\lim_\sigma t^{w}\alpha$ exists as well.

PROPOSITION 1.54. *Consider a variety F with the trivial action of a torus A and an element of the localised K-theory $\alpha \in S_A^{-1}K^A(F)[y]$. Let $w \in \mathrm{Hom}(A,\mathbb{C}^*)\otimes_{\mathbb{Z}}\mathbb{Q}$ be a fractional weight. Let $w_0,...,w_m \in \mathrm{Hom}(A,\mathbb{C}^*)$ be characters such that*

$$w \in \mathrm{conv}(w_0,...,w_m) \subset \mathrm{Hom}(A,\mathbb{C}^*)\otimes_{\mathbb{Z}}\mathbb{R}$$

and w can be expressed as a convex combination of $w_0,...,w_m$ with a nonzero coefficient at w_0. Suppose that for a chosen one parameter subgroup σ all of the limits $\lim_\sigma t^{w_i}\alpha$ exist and we have

$$\lim_\sigma t^{w_0}\alpha = 0\,.$$

Then the limit $\lim_\sigma t^w\alpha$ exists and is equal to zero.

1.3. BB-decomposition

The BB-decomposition was introduced in [**BB73, BB74, BB76**] and further studied in e.g [**CS79, CG83, JS19**] (see [**Car02**] for a survey). We recall its definition and fundamental properties.

DEFINITION 1.55. Let M be a smooth $\mathbb{C}^*$-variety. Let F be a component of the fixed point set $M^{\mathbb{C}^*}$. The positive BB-cell of F is the subset

$$M_F^+ = \{x \in M \mid \lim_{t\to 0} t\cdot x \in F\}\,.$$

Analogously, the negative BB-cell of F is the subset

$$M_F^- = \{x \in M \mid \lim_{t\to \infty} t\cdot x \in F\}\,.$$

It follows from [**BB73**] that

THEOREM 1.56. *Let M be a smooth $\mathbb{C}^*$-variety. Let F be a component of the fixed point set $M^{\mathbb{C}^*}$.*

1. *The BB-cells are locally closed, smooth, algebraic subvarieties of M. Moreover, we have an equality of $\mathbb{C}^*$-vector bundles*
$$T(M_F^+)_{|F} = (TM_{|F})^+ \oplus TF\,.$$
2. *There exists an algebraic morphism*
$$\lim_{t\to 0}\colon M_F^+ \to F\,.$$
3. *Suppose that the variety M is projective. Then, there is a set decomposition (called BB-decomposition)*
$$M = \bigsqcup_{F\subset M^{\mathbb{C}^*}} M_F^+\,.$$
4. *The morphism $\lim_{t\to 0}$ is an affine bundle.*
5. *The BB-decomposition induces a partial order on the set of components of the fixed point set $M^{\mathbb{C}^*}$, defined by the transitive closure of relation*
$$F_2 \cap \overline{M_{F_1}^+} \neq \varnothing \Rightarrow F_1 \geq F_2\,.$$

PROPOSITION 1.57. *Let M be a smooth A-variety. Consider a one parameter subgroup $\sigma\colon \mathbb{C}^* \to A$. Then the BB–cells defined by the $\mathbb{C}^*$-action are A-equivariant subvarieties.*

Definition 1.58 ([**MO19**, Paragraph 3.2.1]). Suppose that M is a smooth A-variety. Consider the vector space of cocharacters

$$\mathfrak{t} := \mathrm{Hom}(\mathbb{C}^*, A) \otimes_{\mathbb{Z}} \mathbb{R}\,.$$

For a fixed point set component $F \subset M^A$, denote by $\nu_1^F, ..., \nu_{\mathrm{codim}\, F}^F$ the torus weights appearing in the normal bundle $\nu(F \subset X)$. A weight chamber is a connected component of the set

$$\mathfrak{t} \setminus \bigcup_{F \subset M^A, i \leq \mathrm{codim} F} \{\nu_i^F = 0\}.$$

Proposition 1.59. *Let M be a smooth projective A-variety. Consider a one parameter subgroup $\sigma \colon \mathbb{C}^* \to A$ such that $\sigma \in \mathfrak{C}$ for some weight chamber $\mathfrak{C}$. Then the fixed point sets M^A and M^σ are equal.*

Proposition 1.60. *Let M be a smooth A-variety. Choose a weight chamber $\mathfrak{C}$. Consider one parameter subgroups σ_1, σ_2 such that $\sigma_1, \sigma_2 \in \mathfrak{C}$. Then σ_1 and σ_2 induce the same decomposition of the normal bundle to the fixed point set into the positive and the negative part. Moreover, the BB-decompositions with respect to these subgroups are equal.*

Proof. The only nontrivial part is the equality of the BB-decompositions. It is a consequence of [**Hu95**, Theorem 3.5]. Alternatively, thanks to the Sumihiro theorem [**Sum74**, Theorem 1] it is enough to prove the proposition for M equal to the projective space. In this case, the proof is straightforward. □

1.4. Stable envelopes

Let A be an algebraic torus and $\mathbb{T} = A \times \mathbb{C}^*$. Suppose that X is a symplectic algebraic $\mathbb{T}$-variety. Suppose that the symplectic form ω is preserved by the torus A and is an eigenvector of the factor $\mathbb{C}^*$. In [**MO19, OS16, Oko17, AO21**] Okounkov and his coauthors defined the stable envelope in the case when the variety X is a symplectic resolution e.g. a Nakajima quiver variety (see [**Bea00, Kal09**] for symplectic resolutions and [**Nak94, Nak98, Gin12**] for an introduction to Nakajima quiver varieties). It was noted in [**Oko21, Kon22**] that one may define the stable envelope in a more general setting. In this section we present the definition of the stable envelope for a cotangent variety with isolated fixed point set. See appendix A for a detailed comparison with the standard definition.

According to [**OS16, Oko17**] the K-theoretical stable envelopes depend on three parameters: polarization $T^{1/2} \in K^{\mathbb{T}}(X)$, one parameter subgroup $\sigma \colon \mathbb{C}^* \to A$ and slope $s \in Pic(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. When X is a cotangent variety T^*M we have a canonical choice of polarization given by TM. In this section we present the definition of the stable envelope only for this polarization (see appendix A for a definition in a general setting). Restriction to a single polarization does not reduce the generality of considerations. The stable envelope for one polarization (and all slopes) determines the stable envelope for any other polarization [**Oko17**, Paragraph 9.1.12].

In the rest of this section we assume that $X = T^*M$ is a cotangent variety with a torus action described in the following example

EXAMPLE 1.61. Consider a smooth projective A-variety M. Suppose that the fixed point set M^A is finite. The cotangent variety $X = T^*M$ is equipped with the action of the torus $\mathbb{T} = A \times \mathbb{C}^*$, such that the action of A is induced from M and the factor $\mathbb{C}^*$ acts on the fibers by scalar multiplication. The fixed point set of this action is finite. Moreover, we have equalities

$$X^{\mathbb{T}} = X^A = M^A.$$

The variety X is equipped with the canonical symplectic nondegenerate form ω. This symplectic structure is compatible with the above action i.e. the form ω is preserved by the torus A and it is an eigenvector of the torus $\mathbb{T}$ with character corresponding to the projection on the second factor. Denote this character by $\mathfrak{h} \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$. For a fixed point $\mathbf{e} \in M^A$ we have an equality of $\mathbb{T}$-representations

$$T_{\mathbf{e}}X = T_{\mathbf{e}}M \oplus (\mathbb{C}_{\mathfrak{h}} \otimes T^*_{\mathbf{e}}M)\,. \tag{1}$$

DEFINITION 1.62. Let M be a smooth projective A-variety. Suppose that the fixed point set M^A is finite. We say that a one parameter subgroup $\sigma\colon \mathbb{C}^* \to A$ is good if and only if $M^A = M^\sigma$.

REMARK 1.63. According to the above definition a one parameter subgroup σ is good if and only if it belongs to some weight chamber (see proposition 1.59).

DEFINITION 1.64. Let M be a smooth projective A-variety. Suppose that the fixed point set M^A is finite. Let $\sigma\colon \mathbb{C}^* \to A$ be a good one parameter subgroup. We say that the pair (M, σ) is admisible if and only if the sum of conormal bundles to BB-cells is a closed subset of the cotangent variety T^*M.

REMARK 1.65. Let $X = T^*M$ be a cotangent variety, described in example 1.61. Let $\sigma\colon \mathbb{C}^* \to A$ be a good one parameter subgroup. The BB–cells of σ in X are the conormal bundles to the BB–cells of σ in M i.e. for $\mathbf{e} \in M^A$

$$X^+_{\mathbf{e}} = \nu^*(M^+_{\mathbf{e}} \subset M)\,. \tag{2}$$

Therefore, the pair (M, σ) is admisible if and only if the sum of BB-cells in X is closed.

REMARK 1.66. To see formula (2) note that $X^+_{\mathbf{e}} \subset T^*M_{|M^+_{\mathbf{e}}}$. The variety $T^*M_{|M^+_{\mathbf{e}}}$ is an affine space with a linear A-action, therefore $X^+_{\mathbf{e}}$ is a linear subspace corresponding to the positive weights.

REMARK 1.67. The pair (M, σ) is admisible if and only if the BB-decomposition of M is a stratification ([**GM88**, Definition on p. 36]) which satisfies Whitney A condition [**Whi65**].

DEFINITION 1.68. Let $X = T^*M$ be a cotangent variety, described in example 1.61. Let $\sigma\colon \mathbb{C}^* \to A$ be a one parameter subgroup such that the pair (M, σ) is admisible. Consider a fractional line bundle $s \in Pic(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ called a slope. The K-theoretic stable envelope is a set of elements $Stab^s(\mathbf{e}) \in K^{\mathbb{T}}(X)$ indexed by the fixed point set M^A, such that

1. Support axiom: For any fixed point $\mathbf{e} \in M^A$ (cf. definition 1.15)

$$\mathrm{supp}(Stab^s(\mathbf{e})) \subset \bigsqcup_{\mathbf{e}' \leq \mathbf{e}} X^+_{\mathbf{e}'}\,.$$

2. Normalization axiom: For any fixed point $\mathbf{e} \in M^A$

$$Stab^s(\mathbf{e})_{|\mathbf{e}} = eu(T^-_{\mathbf{e}}X)\frac{(-1)^{\operatorname{rk} T^+_{\mathbf{e}} M}}{\det T^+_{\mathbf{e}} M}\,.$$

3. Newton inclusion property: Choose any A-linearisation of the slope s. For a pair of fixed points $\mathbf{e}', \mathbf{e} \in M^A$ such that $\mathbf{e}' \leq \mathbf{e}$ we have a containment of the Newton polytopes

$$\mathcal{N}^{\mathrm{A}}\left(Stab^s(\mathbf{e})_{|\mathbf{e}'}\right) + \mathrm{w}_{\mathbf{e}}(s) \subset \mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}'}X)) - \mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M) + \mathrm{w}_{\mathbf{e}'}(s)\,.$$

4. Distinguished point: Choose any A-linearisation of the slope s. For a pair of fixed points $\mathbf{e}', \mathbf{e} \in M^A$ such that $\mathbf{e}' < \mathbf{e}$ the point

$$\mathrm{w}_{\mathbf{e}'}(s) - \mathrm{w}_{\mathbf{e}}(s) - \mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M) \in \operatorname{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}$$

does not belong to the Newton polytope $\mathcal{N}^{\mathrm{A}}\left(Stab^s(\mathbf{e})_{|\mathbf{e}'}\right)$.

Remark 1.69. The order $<$ on the fixed point set M^A is the BB-order according to the one parameter subgroup σ (see theorem 1.56 (5)).

Remark 1.70. The Newton inclusion property may be stated in the equivalent form:

$$\mathcal{N}^{\mathrm{A}}\left(Stab^s(\mathbf{e})_{|\mathbf{e}'}\right) + \mathrm{w}_{\mathbf{e}}(s) - \mathrm{w}_{\mathbf{e}'}(s) \subset \mathcal{N}^{\mathrm{A}}(Stab^s(\mathbf{e}')_{|\mathbf{e}'})\,.$$

Remark 1.71. To state the support axiom of the stable envelope one needs to assume that the subset $\bigsqcup_{\mathbf{e}' \leq \mathbf{e}} X^+_{\mathbf{e}'} \subset X$ is closed. Due to filtrability [**BB76**, Theorem 3] of the BB-decomposition of M, one needs only to assume that the set $\bigsqcup_{\mathbf{e} \in M^A} X^+_{\mathbf{e}} \subset X$ is closed i.e. that the pair (M, σ) is admisible.

Remark 1.72. The above definition may be generalized to a class of varieties broader than cotangent bundles (see definition 7.6). When X is a symplectic resolution such that the fixed point set X^A is finite and the slope s is general enough the above definition is equivalent to Okounkov's definition (after some normalization see proposition 7.10). Okounkov's class is defined only for a general enough slope. In this case the fourth axiom is redundant. In general, the fourth axiom is necessary to obtain uniqueness of the stable envelope (see example 7.11). The above axioms define a K-theory class for an arbitrary slope, which coincides with the class from [**Oko17, OS16**] for a general enough slope.

Proposition 1.73. *There exists at most one class satisfying the axioms of the stable envelope.*

See appendix A for the proof of a more general result (proposition 7.12). The question whether there is a class that satisfies the axioms is difficult. It was proved in [**AO21**] that if X is a Nakajima quiver variety, then the answer is positive.

1.5. Motivic Chern class

The motivic Chern class was defined in [**BSY10**] (see also [**SY07**] for a survey). Its equivariant version is due to [**AMSS19, FRW21**].

Definition 1.74 (after [**AMSS19**, Section 4], see also [**Loo02, Bit04, Bit05**]). Let Y be a quasiprojective A-variety. The group $K^A(Var/Y)$ is the free abelian group generated by symbols $[f\colon Z \to Y]$ for isomorphism classes of A-equivariant morphisms

$f\colon Z \to Y$, where Z is a quasi-projective A-variety, modulo the usual additivity relations

$$[f\colon Z \to Y] = [f\colon U \to Y] + [f\colon Z \setminus U \to Y]$$

for $U \subset Z$ an open invariant subvariety. For every equivariant morphism $g\colon Y \to Y'$ of quasi-projective A-varieties there is a functorial push-forward

$$K^A(Var/Y) \to K^A(Var/Y')$$

given by composition.

DEFINITION 1.75 ([**AMSS19**, Theorem 4.2]). Let $\mathcal{C}$ be the category of quasiprojective A-varieties with proper equivariant maps. There is a unique natural transformation mC_y^A of functors from $\mathcal{C}$ to abelian groups

$$\mathrm{mC}_y^A\colon K^A(Var/-) \longrightarrow G^A(-)[y]$$

such that for a smooth A-variety M we have

$$\mathrm{mC}_y^A(id_M) = \lambda_y(T^*M)\,.$$

This transformation is called the A-equivariant motivic Chern class.

The above definition can be restated in an equivalent way.

DEFINITION 1.76 (after [**FRW21**, Section 2.3]). The motivic Chern class assigns to every A-equivariant map of quasiprojective A-varieties $f\colon Z \to Y$ an element

$$\mathrm{mC}_y^A(f) = \mathrm{mC}_y^A(Z \xrightarrow{f} Y) \in G^A(Y)[y]$$

such that the following properties are satisfied

1. **Additivity:** Let Z be an A-variety and $U \subset Z$ an invariant open subvariety. Then
$$\mathrm{mC}_y^A(Z \xrightarrow{f} Y) = \mathrm{mC}_y^A(U \xrightarrow{f_{|U}} Y) + \mathrm{mC}_y^A(Z \setminus U \xrightarrow{f_{|Z\setminus U}} Y)\,.$$
2. **Functoriality:** For an equivariant proper map $g\colon Y \to Y'$ we have
$$\mathrm{mC}_y^A(Z \xrightarrow{g\circ f} Y') = g_*\,\mathrm{mC}_y^A(Z \xrightarrow{f} Y) \in G^A(Y')[y]\,.$$
3. **Normalization:** For a smooth A-variety M we have
$$\mathrm{mC}_y^A(id_M) = \lambda_y(T^*M) = \sum_{i=0}^{\mathrm{rk}\,T^*M} [\Lambda^i T^*M] y^i \in G^A(M)[y]\,.$$

The equivariant motivic Chern class is the unique assignment satisfying the above properties. For a smooth A-variety M we may consider the class $\mathrm{mC}_y^A(Z \to M)$ as an element of $K^A(M)[y]$ due to Poincaré duality (see theorem 1.6).

The above definition is meaningful also in the non-equivariant setting, i.e. for A equal to the trivial group (see [**BSY10**]).

EXAMPLE 1.77 ([**Web16**, Theorem 7.2]). Let X be a quasiprojective A-variety. We have an equality

$$\mathrm{mC}_y^A(X \to pt) = \chi_y(X)\,,$$

where the class χ_y is the Hirzebruch genus (cf. [**Hir56, BSY10**]).

Let us state some important facts concerning the motivic Chern class

THEOREM 1.78 (Verdier-Riemann-Roch formula [**AMSS19**, Theorem 4.2 (4)]). *Let M and N be smooth quasi-projective A-varieties. Let $\pi\colon N \to M$ be a smooth, A-equivariant map. For any A-equivariant map $f\colon Z \to M$ the following holds:*

$$\lambda_y(T^*_\pi) \cdot \pi^* \,\mathrm{mC}^A_y(Z \xrightarrow{f} M) = \mathrm{mC}^A_y(Z \times_M N \xrightarrow{\pi^* f} N) \in K^A(N)[y]\,.$$

Where T_π denotes the relative tangent bundle to π.

COROLLARY 1.79. *Let M be a smooth A-variety and $U \subset M$ an invariant open subset. For any A-equivariant map $f\colon Z \to M$ we have*

$$\mathrm{mC}^A_y(Z \xrightarrow{f} M)_{|U} = \mathrm{mC}^A_y(f^{-1}(U) \to U) \in K^A(U)[y]\,.$$

PROPOSITION 1.80. [**AMSS19**, Theorem 4.2 (3)] *Let M and N be smooth A-varieties. Consider A-equivariant maps $f\colon X \to M$ and $g\colon Y \to N$. Then*

$$\mathrm{mC}^A_y(X \xrightarrow{f} M) \boxtimes \mathrm{mC}^A_y(Y \xrightarrow{g} N) = \mathrm{mC}^A_y(X \times Y \xrightarrow{f\times g} M \times N)\,.$$

THEOREM 1.81 ([**FRW21**, Theorem 4.2], see also [**Web17**, Theorem 10] and [**Kon21**, Theorem 4.4]). *Let M be a smooth quasiprojective A-variety and $F \subset M^A$ a component of the fixed point set. Let Z be a quasiprojective A-variety and $f\colon Z \to M$ an equivariant map. Then for almost all one parameter subgroups $\sigma\colon \mathbb{C}^* \to A$ (i.e. the set of exceptions is contained in a finite union of hyperplanes) we have*

$$\lim_{\sigma} \left(\frac{\mathrm{mC}^A_y(Z \xrightarrow{f} M)_{|F}}{eu(F \subset M)} \right) = \mathrm{mC}_y(f^{-1}(M^+_F) \xrightarrow{f} F) \in K^A(F)[y]\,.$$

Where $\lim_\sigma$ is the map defined in section 1.2.2.

The above theorem and proposition 1.43 are our main tools for comparing Newton polytopes. Combining these results we obtain the following corollary.

COROLLARY 1.82. *Consider the situation as in the above theorem. Suppose that $F = \mathbf{e}$ is an isolated fixed point. Then*

$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}^A_y(Y \xrightarrow{f} M)_{|\mathbf{e}}) \subset \mathcal{N}^{\mathrm{A}}(eu(\mathbf{e} \subset M)) = \mathcal{N}^{\mathrm{A}}(eu(T_{\mathbf{e}}M))\,.$$

PROPOSITION 1.83. *Let M be a smooth A-variety and $D = \bigcup_{i=1}^n D_i$ an invariant SNC divisor. Consider a subset $I \subset \{1, 2, ..., n\}$. Let*

$$M^o = M \setminus \bigcup_{i=1}^{n} D_i \qquad D_I = \bigcap_{i\in I} D_i \qquad D^o_I = D_I \setminus \bigcup_{i\notin I} D_i\,.$$

Then

$$\mathrm{mC}^A_y(M^o \subset M)_{|D_I} = (1+y)^{|I|} \cdot \mathrm{mC}^A_y(D^o_I \subset D_I) \cdot \prod_{i\in I} \mathcal{O}_M(-D_i)_{|D_I}\,.$$

For the proof we need several lemmas.

LEMMA 1.84. *Let M be a smooth A-variety and D a smooth invariant subvariety of codimension one. Then*

$$\begin{aligned}
\mathrm{mC}^A_y(M \to M)_{|D} &= \mathrm{mC}^A_y(id_D) \cdot (1 + y\mathcal{O}_M(-D)_{|D})\,,\\
\mathrm{mC}^A_y(D \subset M)_{|D} &= \mathrm{mC}^A_y(id_D) \cdot (1 - \mathcal{O}_M(-D)_{|D})\,,\\
\mathrm{mC}^A_y(M \setminus D \subset M)_{|D} &= \mathrm{mC}^A_y(id_D) \cdot (1+y) \cdot \mathcal{O}_M(-D)_{|D}\,.
\end{aligned}$$

Proof. The first equality follows from a short exact sequence

$$0 \longrightarrow \mathcal{O}_M(-D)_{|D} \longrightarrow T^*M_{|D} \longrightarrow T^*D \longrightarrow 0\,.$$

The second from proposition 1.18. The third is a difference between the first and the second. ☐

Lemma 1.85. *Let M be a smooth A-variety and $D = \bigcup_{i=1}^n D_i$ an invariant SNC divisor. Consider a subset $I \subset \{1, 2, ..., n\}$. Then*

$$\mathrm{mC}_y^A(D_I \subset M)_{|D_1} = \begin{cases} \mathrm{mC}_y^A(D_I \subset D_1) \cdot (1 - \mathcal{O}_M(-D_1)_{|D_1}) \text{ if } 1 \in I\,, \\ \mathrm{mC}_y^A(D_{I\cup 1} \subset D_1) \cdot (1 + y\mathcal{O}_M(-D_1)_{|D_1}) \text{ if } 1 \notin I\,. \end{cases}$$

Proof. In the case $1 \in I$, the lemma follows from proposition 1.18. Suppose that $1 \notin I$. Then, we have a pullback square

$$\begin{array}{ccc} D_{I\cup 1} & \xrightarrow{i'} & D_1 \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ D_I & \xrightarrow{i} & M \end{array}$$

which satisfies the assumptions of proposition 1.9 (2). Therefore

$$\begin{aligned} \mathrm{mC}_y^A(D_I \subset M)_{|D_1} &= j^*i_*\,\mathrm{mC}_y^A(id_{D_I}) = i'_*j'^*\,\mathrm{mC}_y^A(id_{D_I}) \\ &= i'_*\left(\mathrm{mC}_y^A(id_{D_{I\cup 1}}) \cdot (1 + y\mathcal{O}_M(-D_1)_{|D_{I\cup\{1\}}})\right) \\ &= \mathrm{mC}_y^A(D_{I\cup 1} \subset D_1) \cdot (1 + y\mathcal{O}_M(-D_1)_{|D_1})\,, \end{aligned}$$

where the third equality follows from lemma 1.84 for $M = D_I$ and $D = D_{I\cup 1}$ and the fourth from the projection formula. ☐

Lemma 1.86. *Let M be a smooth A-variety and $D = \bigcup_{i=1}^n D_i$ an invariant SNC divisor. Then*

$$\mathrm{mC}_y^A(M^o \subset M)_{|D_1} = (1 + y) \cdot \mathrm{mC}_y^A(D_1^o \subset D_1) \cdot \mathcal{O}_M(-D_1)_{|D_1}\,.$$

Proof. The inclusion–exclusion formula implies that

$$\begin{aligned} \mathrm{mC}_y^A(M^o \subset M)_{|D_1} &= \sum_{I\subset\{1,\dots,n\}} (-1)^{|I|} mC(D_I \subset M)_{|D_1} \\ &= \sum_{1\in I} (-1)^{|I|}\,\mathrm{mC}_y^A(D_I \subset M)_{|D_1} + \sum_{1\notin I} (-1)^{|I|}\,\mathrm{mC}_y^A(D_I \subset M)_{|D_1} \\ &= \sum_{I\subset\{2,\dots,n\}} (-1)^{|I|}\,\mathrm{mC}_y^A(D_I \subset M)_{|D_1} \cdot (1+y) \cdot \mathcal{O}_M(-D_1)_{|D_1} \\ &= \mathrm{mC}_y^A(D_1^o \subset D_1) \cdot (1+y) \cdot \mathcal{O}_M(-D_1)_{|D_1}\,. \end{aligned}$$

Where the third equality is a consequence of lemma 1.85. ☐

Proof of proposition 1.83. We proceed by induction on $|I|$. For $|I| = 0$ the proposition is trivial. It is enough to prove that if it holds for some subset I then it holds also for $I \cup x$ for a given $x \notin I$. Without loss of generality we assume that $x = 1$ (and $1 \notin I$). By the inductive assumption we have

$$\mathrm{mC}_y^A(M^o \subset M)_{|D_I} = (1+y)^{|I|} \cdot \mathrm{mC}_y^A(D_I^o \subset D_I) \cdot \prod_{i \in I} \mathcal{O}_M(-D_i)_{|D_I} .$$

Lemma 1.86 for a variety D_I and divisor $\bigcup_{i \notin I} D_i \cap D_I$ implies that

$$\mathrm{mC}_y^A(D_I^o \subset D_I)_{|D_{I\cup 1}} = (1+y) \cdot \mathrm{mC}_y^A(D_{I\cup 1}^o \subset D_{I\cup 1}) \cdot \mathcal{O}_M(-D_1)_{|D_{I\cup 1}} .$$

The two above equations imply that the proposition holds for $I \cup 1$. □

COROLLARY 1.87. *For a smooth invariant subvariety $F \subset D_I$ we have*

$$\mathrm{mC}_y^A\left(M^o \subset M\right)_{|F} \prod_{j \in I} \mathcal{O}_Y(D_j)_{|F} = (1+y)^{|I|}\, \mathrm{mC}_y^A\left(D_I^o \subset D_I\right)_{|F} .$$

At the end of this section, we present some examples of computations.

EXAMPLE 1.88 ([**FRW21**, Subsection 2.7]). Consider the affine line $\mathbb{C}$ with the standard $\mathbb{C}^*$-action. Let $t \in K^{\mathbb{C}^*}(pt)$ be a class corresponding to this representation. Inclusion $i\colon \{0\} \hookrightarrow \mathbb{C}$ induces an isomorphism

$$i^*\colon K^{\mathbb{C}^*}(\mathbb{C}) \to K^{\mathbb{C}^*}(pt).$$

By the normalization property

$$i^*\, \mathrm{mC}_y^A(\mathbb{C} \subset \mathbb{C}) = 1 + y/t .$$

Moreover, using functoriality and proposition 1.18 we get

$$i^*\, \mathrm{mC}_y^A(\{0\} \subset \mathbb{C}) = i^* i_*(1) = 1 - 1/t .$$

The additivity property implies that

$$i^*\, \mathrm{mC}_y^A(\mathbb{C} - \{0\} \subset \mathbb{C}) = (1 + y/t) - (1 - 1/t) = (1+y)/t .$$

More generally, consider a torus A and the affine line $\mathbb{C}$ equipped with a linear A-action. Let $\alpha \in K^A(pt)$ be a class corresponding to this representation. Inclusion $i\colon \{0\} \hookrightarrow \mathbb{C}$ induces an isomorphism $i^*\colon K^A(\mathbb{C}) \to K^A(pt)$. Then,

$$i^*\, \mathrm{mC}_y^A(\{0\} \subset \mathbb{C}) = 1 - 1/\alpha, \qquad i^*\, \mathrm{mC}_y^A(\mathbb{C} \subset \mathbb{C}) = 1 + y/\alpha,$$

$$i^*\, \mathrm{mC}_y^A(\mathbb{C} - \{0\} \subset \mathbb{C}) = (1 + y/\alpha) - (1 - 1/\alpha) = (1+y)/\alpha.$$

EXAMPLE 1.89. Let M be a quasiprojective smooth A-variety. Let $i\colon N \hookrightarrow M$ be an inclusion of an invariant closed smooth subvariety. Then

$$\mathrm{mC}_y^A(N \subset M)_{|N} = i^* i_* \lambda_y(T^*N) = \lambda_y(T^*N) \cdot \lambda_{-1}(\nu^*(N \subset M)) . \tag{3}$$

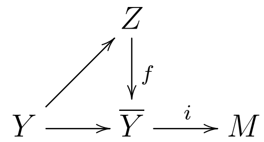
Moreover

$$\begin{aligned}\mathrm{mC}_y^A(N \subset M) =& i_*\, \mathrm{mC}_y^A(id_N) = i_* \lambda_y(T^*N)\\ =& i_*[\mathcal{O}_N] + y i_*[T^*N] + y^2 i_*[\Lambda^2 T^*N] + ... + y^{\dim N} i_*[\det T^*N] .\end{aligned}$$

Thus, the motivic Chern class $\mathrm{mC}_y^A(N \subset M)$ may be considered as a y-deformation of the element $i_*[\mathcal{O}_N]$, which is the fundamental class of N in M. This point of view is pursued in [**Rim21**].

Example 1.90. Let M be a quasiprojective smooth A-variety. Let $i\colon Y \hookrightarrow M$ be an inclusion of an invariant locally closed smooth subvariety. Let $\overline{Y}$ be the closure of Y and $\mathbf{e} \in \overline{Y}$ an isolated fixed point. Suppose that there is a map $f\colon Z \to Y$ such that:

- Z is a smooth A-variety.
- The map f is surjective, proper and A-equivariant.
- The subvariety $\partial Z = f^{-1}(\overline{Y} \setminus Y)$ is a simple normal crossing divisor.
- The restriction $f_{|f^{-1}(Y)}\colon f^{-1}(Y) \to Y$ is an isomorphism.



Later, we will call such a map a SNC resolution of singularities (see definition 3.1). Let $\partial Z = \bigcup_{i=1}^m D_i$. For a subset $I \subset \{1, ..., m\}$ let $D_I = \bigcap_{i\in I} D_i$. It follows that

$$\begin{aligned}
\mathrm{mC}_y^A(Y \longrightarrow M) =& i_* \,\mathrm{mC}_y^A(Y \longrightarrow \overline{Y}) \\
=& f_* i_* \,\mathrm{mC}_y^A(Y \longrightarrow Z) \\
=& (i \circ f)_* \,\mathrm{mC}_y^A\left(Z \setminus \bigcup D_i \subset Z\right) \\
=& (i \circ f)_* \sum_{I \subset \{1,...,m\}} (-1)^{|I|}\, \mathrm{mC}_y^A\left(D_I \subset Z\right) .
\end{aligned}$$

Suppose that the preimage $f^{-1}(\mathbf{e})$ is a finite set. The Lefschetz-Riemann-Roch formula (theorem 1.24) implies that

$$\text{(4)}\quad \mathrm{mC}_y^A(Y \longrightarrow M)_{|\mathbf{e}} = \sum_{\mathbf{e}' \in f^{-1}(\mathbf{e})} \left(\frac{\lambda_{-1}(T^*_{\mathbf{e}} M)}{\lambda_{-1}(T^*_{\mathbf{e}'} Z)} \sum_{I \subset \{1,...,m\}} (-1)^{|I|}\, \mathrm{mC}_y^A\left(D_I \subset Z\right)_{|\mathbf{e}'} \right) .$$

Consider a point $\mathbf{e}' \in f^{-1}(\mathbf{e})$. Let $I_{\mathbf{e}'} = \{i \in \{1, ..., m\} | \mathbf{e}' \in D_i\}$. Let

$$d_i = \mathcal{O}(D_i)_{|\mathbf{e}'} \in K^A(\mathbf{e}')$$

be a weight of normal space to D_i at $\mathbf{e}'$. Divisors $\{D_i\}_{i \in I_{\mathbf{e}'}}$ are SNC, thus their normal spaces at $\mathbf{e}$ sum up to a linear space of dimension $|I_{\mathbf{e}'}|$. Denote by $\alpha_1, ..., \alpha_{\dim Z - |I_{\mathbf{e}'}|}$ the remaining weights in the tangent space $T_{\mathbf{e}'} Z$. The class $\mathrm{mC}_y^A(D_I \hookrightarrow Z)_{|\mathbf{e}'}$ may be nonzero only when $I \subset I_{\mathbf{e}'}$. For such I we have

$$\begin{aligned}
\mathrm{mC}_y^A\left(D_I \subset Z\right)_{|\mathbf{e}'} =& \lambda_{-1}(\nu^*_{\mathbf{e}'}(D_I \subset Z)) \cdot \lambda_y(T^*_{\mathbf{e}'} D_I) \\
=& \lambda_y(T^*_{\mathbf{e}'} D_{I_{\mathbf{e}'}}) \cdot \lambda_{-1}(\nu^*_{\mathbf{e}'}(D_I \subset Z)) \cdot \lambda_y(\nu^*_{\mathbf{e}'}(D_{I_{\mathbf{e}'}} \subset D_I)) \\
=& \prod_{i=1}^{\dim Z - |I_{\mathbf{e}'}|} \left(1 + \frac{y}{\alpha_i}\right) \cdot \prod_{i \in I} \left(1 + \frac{y}{d_i}\right) \cdot \prod_{i \in I_{\mathbf{e}'} \setminus I} \left(1 - \frac{1}{d_i}\right) .
\end{aligned}$$

Therefore, the formula (4) simplifies

$$\begin{aligned}\sum_{I\subset\{1,\dots,m\}}(-1)^{|I|}\,\mathrm{mC}_y^A\,(D_I\subset Z)_{|\mathbf{e}'} &= \sum_{I\subset I_{\mathbf{e}'}}(-1)^{|I|}\,\mathrm{mC}_y^A\,(D_I\subset Z)_{|\mathbf{e}'}\\ &= \prod_{i=1}^{\dim Z-|I_{\mathbf{e}'}|}\left(1+\frac{y}{\alpha_i}\right)\cdot\prod_{i\in I_{\mathbf{e}'}}\left(\frac{1+y}{d_i}\right)\end{aligned}$$

If the fixed point $\mathbf{e}$ lies in Y than we have $|f^{-1}(\mathbf{e})|=1$ and $I_{\mathbf{e}}=\varnothing$. Therefore, we recover the formula from example 1.89.

Remark 1.91. Consider a locally closed smooth subvariety $Y\subset M$ and SNC resolution $Z\to\overline{Y}$ as in the above example. Suppose that the fixed point set Z^A is finite. The above example implies that we may compute the motivic Chern class $\mathrm{mC}_y^A(Y\subset M)$ using only the calculus of rational functions.

CHAPTER 2

Motivic Chern class as stable envelope

2.1. Statement of result

Let A be an algebraic torus. Let M be a smooth, projective A-variety. Suppose that the fixed point set M^A is finite. Consider the induced action of the product torus $\mathbb{T} = A \times \mathbb{C}^*$ on the cotangent variety $X = T^*M$ such as in example 1.61. Denote by π the projection

$$\pi \colon T^*M \to M.$$

Let $\sigma \colon \mathbb{C}^* \to A$ be a good one parameter subgroup (cf. definition 1.62).

In this chapter we consider the stable envelope for the cotangent variety X, the trivial slope θ and the one parameter subgroup σ (see definition 1.68). Our aim is to prove that after a suitable normalization the motivic Chern class of BB-cell satisfies all but one of the axioms of the stable envelope.

Definition 2.1. Suppose that Y is a $\mathbb{T}$-variety, such that the factor $\mathbb{C}^*$ acts trivially. Let ρ be the map

$$\rho \colon K^A(Y)[y] \to K^{\mathbb{T}}(Y)$$

given by $\rho(y) = -\mathfrak{h}$.

Theorem 2.2. *Let* $\mathbf{e} \in M^A$ *be a fixed point. The class*

$$\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)) \in K^{\mathbb{T}}(X)$$

satisfies the normalization axiom and the Newton inclusion property of the stable envelope $Stab^\theta(\mathbf{e})$*. Moreover, it satisfies the distinguished point axiom.*

The rest of this chapter is devoted to the proof of the above theorem. This theorem is a direct consequence of propositions 2.5, 2.6 and 2.9.

Remark 2.3. The map ρ commutes with pullbacks. Thus, for an arbitrary fixed point $\mathbf{e}' \in M^A$ there is an equality

$$\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M))_{|\mathbf{e}'} = \rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)_{|\mathbf{e}'}) \in K^{\mathbb{T}}(\mathbf{e}')$$

Remark 2.4. In this chapter we do not assume any regularity conditions on the BB-decomposition of M. In particular we do not assume that the pair (M, σ) is admisible (definition 1.64). As noted in remark 1.71 the support axiom cannot be stated on this level of generality.

2.2. Normalization axiom

Proposition 2.5. *Consider a fixed point* $\mathbf{e} \in M^A$*. The class*

$$\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)) \in K^{\mathbb{T}}(X)$$

satisfies the normalization axiom of the stable envelope $Stab^\theta(\mathbf{e})$. Namely

$$\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M))_{|\mathbf{e}} = (-1)^{\dim(M^+_{\mathbf{e}})}\frac{eu(T^-_{\mathbf{e}}X)}{\det(T^+_{\mathbf{e}}M)}$$

Proof. Let i denote inclusion of the closure of the BB-cell

$$i\colon \overline{M^+_{\mathbf{e}}}\to M.$$

The center of the cell $\mathbf{e}\in\overline{M^+_{\mathbf{e}}}$ is a smooth point. Thus, the motivic Chern class localized at $\mathbf{e}$ is equal to (cf. formula (3))

$$\begin{aligned}\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)_{|\mathbf{e}} &= \left(i^*i_*\,\mathrm{mC}^A_y\left(M^+_{\mathbf{e}}\to \overline{M^+_{\mathbf{e}}}\right)\right)_{|\mathbf{e}}\\ &= eu(T^-_{\mathbf{e}}M)\cdot \mathrm{mC}^A_y\left(id_{M^+_{\mathbf{e}}}\right)_{|\mathbf{e}}\\ &= \lambda_{-1}((T^-_{\mathbf{e}}M)^*)\cdot\lambda_y((T^+_{\mathbf{e}}M)^*)\in K^A(M)[y]\,.\end{aligned}$$

Equality (1) of $\mathbb{T}$-representation implies that:

$$T^-_{\mathbf{e}}X = T^-_{\mathbf{e}}M\oplus(\mathbb{C}_{\mathfrak{h}}\otimes(T^+_{\mathbf{e}}M)^*)\,.$$

It follows that

$$eu(T^-_{\mathbf{e}}X) = \lambda_{-1}((T^-_{\mathbf{e}}M)^*)\cdot\lambda_{-1}(\mathbb{C}_{1/\mathfrak{h}}\otimes T^+_{\mathbf{e}}M)\in K^{\mathbb{T}}(\mathbf{e})\,.$$

We apply proposition 1.17 for $E=\mathbb{C}_{1/\mathfrak{h}}\otimes T^+_{\mathbf{e}}M$ and obtain

$$\begin{aligned}eu(T^-_{\mathbf{e}}X) =&\lambda_{-1}(\mathbb{C}_{1/\mathfrak{h}}\otimes T^+_{\mathbf{e}}M)\cdot\lambda_{-1}((T^-_{\mathbf{e}}M)^*)\\ =&(-1)^{\dim(M^+_{\mathbf{e}})}\cdot\det(\mathbb{C}_{1/\mathfrak{h}}\otimes T^+_{\mathbf{e}}M)\cdot\lambda_{-1}(\mathbb{C}_{\mathfrak{h}}\otimes(T^+_{\mathbf{e}}M)^*)\cdot\lambda_{-1}((T^-_{\mathbf{e}}M)^*)\\ =&(-\mathfrak{h})^{\dim(M^+_{\mathbf{e}})}\cdot\det(T^+_{\mathbf{e}}M)\cdot\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*)\cdot\lambda_{-1}((T^-_{\mathbf{e}}M)^*)\,.\end{aligned}$$

Setting $\rho(y)=-\mathfrak{h}$ we deduce

$$\begin{aligned}(-1)^{\dim(M^+_{\mathbf{e}})}\frac{eu(T^-_{\mathbf{e}}X)}{\det(T^+_{\mathbf{e}}M)} =&\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\cdot\lambda_{-1}((T^-_{\mathbf{e}}M)^*)\cdot\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*)\\ =&\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\rho\left(\lambda_{-1}((T^-_{\mathbf{e}}M)^*)\cdot\lambda_y((T^+_{\mathbf{e}}M)^*)\right)\\ =&\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\rho\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)\right)_{|\mathbf{e}}\\ =&\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)\right)_{|\mathbf{e}}\,.\end{aligned}$$

□

2.3. Newton inclusion property

Proposition 2.6. *Consider a fixed point $\mathbf{e}\in M^A$. The class*

$$\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M))\in K^{\mathbb{T}}(X)$$

satisfies the Newton polytope property of the stable envelope $Stab^\theta(\mathbf{e})$, i.e. for any fixed point $\mathbf{e}'<\mathbf{e}$ we have

$$\mathcal{N}^{\mathrm{A}}\left(\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M))_{|\mathbf{e}'}\right)\subset\mathcal{N}^{\mathrm{A}}\left(eu(T^-_{\mathbf{e}'}X)\right)-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M)$$

Before presenting the proof of the above proposition we prove two technical lemmas.

LEMMA 2.7. *There is an equality*

$$\mathcal{N}^{\mathrm{A}}\left(\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M))_{|\mathbf{e}'}\right)=\mathcal{N}^{\mathrm{A}}\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)_{|\mathbf{e}'}\right)$$

PROOF. It follows from remark 2.3 and example 1.32 that

$$\mathcal{N}^{\mathrm{A}}\left(\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M))_{|\mathbf{e}'}\right)=\mathcal{N}^{\mathrm{A}}\left(\rho\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)_{|\mathbf{e}'}\right)\right)\,.$$

Moreover, the factor $\mathbb{C}^*$ of $\mathbb{T}=A\times\mathbb{C}^*$ acts trivially on M, thus (cf. proposition 1.33)

$$\mathcal{N}^{\mathrm{A}}\left(\rho\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)_{|\mathbf{e}'}\right)\right)=\mathcal{N}^{\mathrm{A}}\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)_{|\mathbf{e}'}\right)\,.$$

□

LEMMA 2.8. *For any fixed point* $\mathbf{e}\in M^A$ *we have*

$$\mathcal{N}^{\mathrm{A}}(eu(T_{\mathbf{e}}M))\subset\mathcal{N}^{\mathrm{A}}\left(eu(T^-_{\mathbf{e}}X)\right)-\mathrm{w}_{\mathbf{e}}(\det T^+_{\mathbf{e}}M)\,.$$

PROOF. There is an equality

$$\mathcal{N}^{\mathrm{A}}\left(eu\left(T_{\mathbf{e}}M\right)\right)=\mathcal{N}^{\mathrm{A}}\left(\lambda_{-1}\left(\left(T^-_{\mathbf{e}}M\right)^*\right)\cdot\lambda_{-1}\left(\left(T^+_{\mathbf{e}}M\right)^*\right)\right)\,.$$

We use proposition 1.28 (e) for a homomorphism $K^{\mathbb{C}^*}(\mathbf{e})\to K^{\mathbb{C}^*}(\mathbf{e})$ which sends $\mathfrak{h}$ to zero and obtain

$$\mathcal{N}^{\mathrm{A}}\left(\lambda_{-1}\left(\left(T^-_{\mathbf{e}}M\right)^*\right)\cdot\lambda_{-1}\left(\left(T^+_{\mathbf{e}}M\right)^*\right)\right)\subset\mathcal{N}^{\mathrm{A}}\left(\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\lambda_{-1}\left(\left(T^-_{\mathbf{e}}M\right)^*\right)\cdot\lambda_{-\mathfrak{h}}\left(\left(T^+_{\mathbf{e}}M\right)^*\right)\right)\,.$$

The proof of proposition 2.5 implies that

$$(-\mathfrak{h})^{\dim(M^+_{\mathbf{e}})}\cdot\lambda_{-1}((T^-_{\mathbf{e}}M)^*)\cdot\lambda_{-\mathfrak{h}}\left(\left(T^+_{\mathbf{e}}M\right)^*\right)=\frac{eu(T^-_{\mathbf{e}}X)}{\det(T^+_{\mathbf{e}}M)}\,.$$

To conclude, we have

$$\mathcal{N}^{\mathrm{A}}\left(eu\left(T_{\mathbf{e}}M\right)\right)\subset\mathcal{N}^{\mathrm{A}}\left(\frac{eu(T^-_{\mathbf{e}}X)}{\det(T^+_{\mathbf{e}}M)}\right)=\mathcal{N}^{\mathrm{A}}\left(eu(T^-_{\mathbf{e}}X)\right)-\mathrm{w}_{\mathbf{e}}(\det T^+_{\mathbf{e}}M)\,,$$

which proves the lemma. □

PROOF OF PROPOSITION 1.28. Due to lemmas 2.7 and 2.8 it is enough to prove that

$$\mathcal{N}^{\mathrm{A}}\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)_{|\mathbf{e}'}\right)\subset\mathcal{N}^{\mathrm{A}}\left(eu(T_{\mathbf{e}'}M)\right)\,.\tag{5}$$

This follows from corollary 1.82. □

2.4. Distinguished point

PROPOSITION 2.9. *For a pair of fixed points* $\mathbf{e}',\mathbf{e}\in M^A$ *such that* $\mathbf{e}>\mathbf{e}'$*, we have*

$$-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M)\notin\mathcal{N}^{\mathrm{A}}\left(\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M))_{|\mathbf{e}'}\right)\,.$$

PROOF. Due to lemma 2.7 it is enough to prove that

$$-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M)\notin\mathcal{N}^{\mathrm{A}}\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)_{|\mathbf{e}'}\right)\,.$$

For any any one parameter subgroup $\sigma'\colon\mathbb{C}^*\to A$ we consider the induced maps

$$\sigma'^*\colon K^A(M)\to K^{\mathbb{C}^*}(M)\,,\qquad\pi_{\sigma'}\colon\ \mathrm{Hom}(A,\mathbb{C}^*)\otimes\mathbb{R}\to\mathrm{Hom}(\mathbb{C}^*,\mathbb{C}^*)\otimes\mathbb{R}\,.$$

It is enough to show that for some one parameter subgroup σ' we have

$$\pi_{\sigma'}(-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M) \notin \pi_{\sigma'}\left(\mathcal{N}^{\mathrm{A}}\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)_{|\mathbf{e}'}\right)\right) .$$

Consider the one parameter group σ which induces the BB-decomposition of M. Then, the point $\pi_\sigma(-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M))$ is the lowest term of the line segment

$$\mathcal{N}^{\mathbb{C}^*}\left(\sigma^* eu(T^-_{\mathbf{e}'}X)\right) .$$

Theorem 1.81 implies that

$$\lim_{\sigma}\left(\frac{\mathrm{mC}^A_y(M^+_{\mathbf{e}} \subset M)_{|\mathbf{e}'}}{eu(T^-_{\mathbf{e}'}X)}\right) = \mathrm{mC}_y(M^+_{\mathbf{e}} \cap M^+_{\mathbf{e}'} \to \mathbf{e}') = \mathrm{mC}_y(\varnothing \to \mathbf{e}') = 0\,.$$

Thus, the lowest term of the line segment $\mathcal{N}^{\mathbb{C}^*}\left(\sigma^*\,\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)_{|\mathbf{e}'}\right)$ is greater than the lowest term of the line segment $\mathcal{N}^{\mathbb{C}^*}\left(\sigma^* eu(T^-_{\mathbf{e}'}X)\right)$ i.e.

$$\pi_\sigma(-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M) \notin \mathcal{N}^{\mathbb{C}^*}\left(\sigma^*\,\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)_{|\mathbf{e}'}\right) .$$

Moreover, we have

$$\mathcal{N}^{\mathbb{C}^*}\left(\sigma^*\,\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)_{|\mathbf{e}'}\right) = \pi_\sigma\left(\mathcal{N}^{\mathrm{A}}\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)_{|\mathbf{e}'}\right)\right) .$$

Therefore

$$\pi_\sigma(-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M) \notin \pi_\sigma\left(\mathcal{N}^{\mathrm{A}}\left(\mathrm{mC}^A_y(M^+_{\mathbf{e}} \to M)_{|\mathbf{e}'}\right)\right) ,$$

which proves the proposition. □

REMARK 2.10. In [**Kon22**, Section 6] it is proved that the motivic Chern class satisfies also the axioms of the stable envelope for a small anti-ample slope. Here we omit this proof. It is a consequence of a more general statement (see corollary 4.3).

CHAPTER 3

Twisted motivic Chern class

3.1. Definition

Let A be an algebraic torus. Let $(W, \partial W)$ be a pair, consisting of an algebraic quasiprojective A-variety W and an invariant closed subvariety $\partial W \subset W$. We assume that $W^o = W \setminus \partial W$ is smooth. We call the subvariety ∂W boundary of W. Let Δ be a $\mathbb{Q}$-Cartier divisor on W with support contained in the boundary $|\Delta| \subset \partial W$.

Definition 3.1. Let $(W, \partial W)$ be as above. Consider an algebraic map

$$f\colon (Y, \partial Y) \to (W, \partial W)$$

such that

- Y is a smooth A-variety
- The map f is surjective, proper and A-equivariant.
- The subvariety $\partial Y = f^{-1}(\partial W)$ is a simple normal crossing divisor.
- Let $Y^o = Y \setminus \partial Y$. The restriction $f_{|Y^o}\colon Y^o \to W^o$ is an isomorphism.

We call such a map a SNC resolution of singularities.

Theorem 3.2 ([**Hir64**]). *Let $(W, \partial W)$ be as above. There exists a SNC resolution of singularities $f\colon (Y, \partial Y) \to (W, \partial W)$.*

Definition 3.3 ([**Laz04**, Definition 9.1.2]). Let $\Delta = \sum q_i D_i$ be a $\mathbb{Q}$-divisor on X. We define the round-up divisor $\lceil D \rceil$ as

$$\lceil D \rceil = \sum \lceil q_i \rceil D_i \,,$$

where the round-up $\lceil q \rceil$ of a rational number q is the smallest integer greater or equal than q, i.e.

$$\lceil q \rceil = \min\{n \in \mathbb{Z} | n \geq q\} \,.$$

Definition 3.4. Let $(W, \partial W)$ and Δ be as above. Let $f\colon (Y, \partial Y) \to (W, \partial W)$ be a SNC resolution of singularities. The twisted motivic Chern class $\mathrm{mC}^A_y(W, \partial W; \Delta)$ is defined by the formula

$$\mathrm{mC}^A_y(X, \partial X; \Delta) = f_*\left(\mathcal{O}_Y(\lceil f^*(\Delta) \rceil) \cdot \mathrm{mC}^A_y(Y^o \subset Y)\right) \in G^A(W)[y] \,.$$

The twisted motivic Chern class is an element of the equivariant K-theory of coherent sheaves tensored with the polynomial ring $\mathbb{Q}[y]$. In the next section we will prove that it does not depend on the choice of SNC resolution. In our application we will study the image of $\mathrm{mC}^A_y(W, \partial W; \Delta)$ in the K-theory of a smooth ambient space. There the K-theory of coherent sheaves is isomorphic to the K-theory of locally free sheaves.

EXAMPLE 3.5. Let $(W, \partial W) = (\mathbb{P}^1, \{0\})$ with the standard $A = \mathbb{C}^*$–action. The tangent character at $0 \in (\mathbb{P}^1)^A$ is equal to t and the character at ∞ is equal to t^{-1}. Let $\Delta = \lambda\{0\}$, where $\lambda \in \mathbb{Q}$. Since $\mathbb{P}^1$ is smooth, we may take $\mathbb{P}^1 = Y$ and $f = id$ as a SNC resolution of singularities. Hence,

$$\mathrm{mC}_y^A(\mathbb{P}^1, \partial\mathbb{P}^1; \lambda\{0\}) = \mathcal{O}_{\mathbb{P}^1}(\lceil\lambda\rceil\{0\}) \cdot \mathrm{mC}_y^A(\mathbb{P}^1 \setminus \{0\} \subset \mathbb{P}^1)\,.$$

The weights of the divisor Δ at the fixed points are

$$\mathrm{w}_0(\Delta) = \lambda\,, \qquad \mathrm{w}_\infty(\Delta) = 0\,.$$

The restrictions of the class $\mathrm{mC}_y^A(\mathbb{P}^1, \partial\mathbb{P}^1; \lambda\{0\})$ to the fixed points are

$$\mathrm{mC}_y^A(\mathbb{P}^1, \partial\mathbb{P}^1; \lambda\{0\})_{|0} = t^{\lceil\lambda\rceil} \cdot (1+y)t^{-1}\,,$$
$$\mathrm{mC}_y^A(\mathbb{P}^1, \partial\mathbb{P}^1; \lambda\{0\})_{|\infty} = 1 + yt\,.$$

REMARK 3.6. Suppose that Δ_1 is an integral Cartier divisor. Then for an arbitrary $\mathbb{Q}$-Cartier divisor Δ_2

$$\mathrm{mC}_y^A(X, \partial X; \Delta_1 + \Delta_2) = \mathcal{O}_X(\Delta_1) \cdot \mathrm{mC}_y^A(X, \partial X; \Delta_2)$$

by the projection formula (proposition 1.8). This agrees with the behaviour of stable envelopes [**AMSS19**, Lemma 8.2c].

PROPOSITION 3.7. *Let Δ_1 be an arbitrary $\mathbb{Q}$-Cartier divisor. Let Δ_2 be an effective $\mathbb{Q}$-Cartier divisor. Suppose that Δ_2 is small enough. Then*

$$\mathrm{mC}_y^A(X, \partial X; \Delta_1 - \Delta_2) = \mathrm{mC}_y^A(X, \partial X; \Delta_1)\,.$$

For example

$$\mathrm{mC}_y^A(X, \partial X; -\Delta_2) = \mathrm{mC}_y^A(X^o \subset X)\,.$$

PROOF. The resolution of singularities f is surjective and birational, therefore the divisor $f^*\Delta_2$ is effective. If the coefficients of Δ_2 are small enough then

$$\lceil f^*\Delta_1 - f^*\Delta_2 \rceil = \lceil f^*\Delta_1 \rceil$$

due to semi-continuity of the ceiling function. □

REMARK 3.8 (About relation with multiplier ideals, [**BL04, Laz04**]). The construction of the twisted motivic Chern class has a lot in common with the definition of multiplier ideals [**Laz04**, Definition 9.2.1]. Let X be a smooth variety of dimension n. Let Δ be an effective $\mathbb{Q}$-divisor and ∂X any subvariety such that $|K_X + \Delta| \subset \partial X$. Then the coefficient of y^n of the class

$$(-1)^n\, \mathrm{mC}_y^A(X, \partial X; -K_X - \Delta)$$

is equal to the K-theory class of the multiplier ideal for Δ.

PROPOSITION 3.9. *Let $(W, \partial W)$ and Δ be such as in definition 3.4. Then*

(1) For an open invariant subvariety $U \subset W$

$$\mathrm{mC}_y^A(U, \partial W \cap U; \Delta \cap U) = \mathrm{mC}_y^A(W, \partial W; \Delta)_{|U} \in G^A(U)[y]\,.$$

(2) For a smooth A-variety S

$$\mathrm{mC}_y^A(W \times S, \partial W \times S; \Delta \times S) = \mathrm{mC}_y^A(id_S) \boxtimes \mathrm{mC}_y^A(W, \partial W; \Delta) \in G^A(W \times S)[y]\,.$$

PROOF. To compute the twisted motivic Chern classes in (i) and (ii) we can use the resolution of $(W, \partial W)$ restricted to U or multiplied by S. □

3.2. Independence from the resolution

To prove that the class given in definition 3.4 does not depend on the choice of resolution we apply the weak factorization theorem [**AKMW02**], [**Wło09**, Theorem 0.0.1]. It is enough to consider two resolutions which differ by a single blow-up in a center contained in the boundary, which has normal intersections (see e.g. [**Kol07**, Definition 3.24]) with components of the boundary divisor.

PROPOSITION 3.10. *Let Y be a smooth A-variety and let $\partial Y = \bigcup_{k=1}^{m} \partial Y_k$ be a SNC divisor. Let $C \subset \partial Y$ be a smooth invariant subvariety such that for each component ∂Y_k either*

(1) C is contained in ∂Y_k,
(2) C intersect ∂Y_k normally.

Let $b\colon Z \to Y$ be the blow-up of Y in C and E the exceptional divisor. Denote maps according to the diagram

$$\begin{array}{ccc} E & \xrightarrow{i} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle b} \\ C & \xrightarrow[i']{} & Y \end{array}$$

Let D be a SNC, $\mathbb{Q}$-Cartier divisor on Y with support contained in the boundary ∂Y. Let $\partial Z = b^{-1}(\partial Y)$ and $Z^o = Z \setminus \partial Z$. Then

$$b_*\left(\mathcal{O}_Z(\lceil b^*D \rceil) \cdot \mathrm{mC}_y^A(Z^o \subset Z)\right) = \mathcal{O}_Y(\lceil D \rceil) \cdot \mathrm{mC}_y^A(Y^o \subset Y).$$

REMARK 3.11. The A-action on the variety Y induces an A-action on the blow up Z.

REMARK 3.12. If

$$\mathcal{O}_Z(\lceil b^*D \rceil) = \mathcal{O}_Z(b^*\lceil D \rceil)$$

then the proposition follows from the projection formula (proposition 1.8).

Before the proof let us state an important corollary.

COROLLARY 3.13. *The twisted motivic Chern class does not depend on the choice of resolution.*

PROOF. Let $(W, \partial W)$ and Δ be such as in definition 3.4. Let

$$f\colon (Y, \partial Y) \to (W, \partial W)$$

be a SNC resolution. By [**AKMW02**], [**Wło09**, Theorem 0.0.1] any two resolutions can be joined by a sequence of blow-ups or blow-downs with centers normally intersecting the components of the boundary divisor. For such a blow-up

$$Z \xrightarrow{b} Y \xrightarrow{f} W\,,$$

we use proposition 3.10 for $D = f^*\Delta$ and acquire

$$f_*\left(\mathcal{O}_Y(\lceil f^*\Delta \rceil) \cdot \mathrm{mC}_y^A(Y^o \subset Y)\right) = (f \circ b)_*\left(\mathcal{O}_Z(\lceil (f \circ b)^*\Delta \rceil) \cdot \mathrm{mC}_y^A(Z^o \subset Z)\right)\,.$$

□

The rest of this section is devoted to the proof of proposition 3.10. Let

$$\partial Z = E \cup \bigcup_{k=1}^{m} \partial Z_i \,,$$

where ∂Z_i is a proper transform of the divisor ∂Y_i. Let

$$D = \sum_{k=1}^{m} c_k \partial Y_k \,.$$

Without loss of generality assume that C is contained in the intersection of the boundary components $\bigcap_{k=1}^{r} \partial Y_k$ and is not contained in any divisor ∂Y_i for $i > r$. Thus, the multiplicity of E in b^*D is equal to $\sum_{k=1}^{r} c_k$. The difference

$$b^*(\lceil D \rceil) - \lceil b^* D \rceil = \left(\sum_{k=1}^{r} \lceil c_k \rceil - \left\lceil \sum_{k=1}^{r} c_k \right\rceil \right) E$$

is equal to sE, where s is a nonnegative integer smaller than r. We need to compute

$$\begin{aligned}
&b_*\Big(\mathcal{O}_Z(\lceil b^* D \rceil) \cdot \mathrm{mC}_y^A(Z^o \subset Z)\Big) = \\
&b_*\Big(\mathcal{O}_Z(b^*\lceil D \rceil - (b^*\lceil D \rceil - \lceil b^* D \rceil)) \cdot \mathrm{mC}_y^A(Z^o \subset Z)\Big) = \\
&b_*\Big(\mathcal{O}_Z(b^*\lceil D \rceil) \cdot \mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^o \subset Z)\Big) = \\
&\mathcal{O}_Y(\lceil D \rceil) \cdot b_*\Big(\mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^o \subset Z)\Big) .
\end{aligned}$$

It is enough to prove that for $s \in \{0, 1, ..., r-1\}$

$$b_*\Big(\mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^o \subset Z)\Big) = \mathrm{mC}_y^A(Y^o \subset Y) \,.$$

First, we consider the case when $C \cap \partial Y_i = \varnothing$ for $i > r$. We need a technical lemma.

Lemma 3.14. *Let C be a smooth algebraic A-variety. Let $V \to C$ be an A-vector bundle with a collection $\{V_k\}_{k=1}^r$ of codimension one invariant subbundles $V_k \subset V$. Suppose that $r \leq \operatorname{rk} V$ and the bundles V_k are in general position at each point of C. Consider the projective bundle $g\colon E = \mathbb{P}(V) \to C$ and the divisors $B_k = \mathbb{P}(V_k)$. Let $B = \bigcup_{k=1}^{r} B_k$. Then*

$$g_*\left(\mathcal{O}_{E/C}(s) \cdot \mathrm{mC}_y^A(E \setminus B \subset E)\right) = 0 \qquad \textit{for } s \in \{1, 2, \dots, r-1\} \,.$$

Proof. Let W be an arbitrary A-vector bundle over C. Let $g_W : \mathbb{P}(W) \to C$ be the associated projective bundle and $\Omega^1_{\mathbb{P}(W)/C} \in Vect^A(\mathbb{P}(W))$ the relative cotangent bundle. We apply the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}(W)/C} \to g_W^* W^*(-1) \to \mathcal{O}_{\mathbb{P}(W)} \to 0 \,, \tag{6}$$

where the twist (-1) denotes the tensor product with the relative $\mathcal{O}_{\mathbb{P}(W)/C}(-1)$ bundle. It follows that the motivic Chern class of $id_{\mathbb{P}(W)}$ can be presented as

$$\begin{aligned}
\mathrm{mC}_y^A(id_{\mathbb{P}(W)}) =& g_W^*(\mathrm{mC}_y^A(id_C)) \cdot \lambda_y(\Omega^1_{\mathbb{P}(W)/C}) \\
=& g_W^*(\mathrm{mC}_y^A(id_C)) \cdot \lambda_y(g^* W^*(-1))/(1+y) \in K^A(\mathbb{P}(W))[y] \,.
\end{aligned}$$

The first equality follows from the Verdier-Riemann-Roch theorem 1.78 and the second from the exact sequence (6).
For any subset $I \subset \{1, 2, \dots, r\}$ let

$$V_I = \bigcap_{i \in I} V_i \text{ and } B_I = \mathbb{P}(V_I)\,.$$

The vector bundles V_i are in general position, thus for an arbitrary I the variety V_I is also a vector bundle over C. Denote by $g_I = g_{|B_I} \colon B_I \to C$ projections and by $i_I \colon B_I \hookrightarrow E$ inclusions. Consider vector bundles

$$L_k = \ker(V^* \to V_k^*)\,.$$

The bundles V_k are in general position, so $\oplus_{k=1}^r L_k$ is a subbundle of V^*. Let $K = V^* / \oplus_{k=1}^r L_k$. Thus, for any subset I

$$V_I^* = K + \sum_{k \notin I} L_k \in K^A(C)\,.$$

Previous discussion for $W = V_I$ implies that in $K^A(B_I)[y]$ we have an equality

$$\begin{aligned}(1+y)\,\mathrm{mC}_y^A(id_{B_I}) =& g_I^*\,\mathrm{mC}_y^A(id_C)\cdot\lambda_y(g_I^*V_I^*(-1))\\ =& g_I^*\,\mathrm{mC}_y^A(id_C)\cdot\left(\prod_{k\notin I} 1 + y g_I^* L_k(-1)\right)\cdot\lambda_y(g_I^* K^*(-1))\,.\end{aligned}$$

By definition $\mathcal{O}_E(B_k) = g^* L_k^*(1)$. Moreover the divisor B is SNC, so proposition 1.19 implies

$$i_{I*}(1) = [\mathcal{O}_{B_I}] = \prod_{k\in I} 1 - \mathcal{O}_E(-B_k) = \prod_{k\in I} 1 - g^* L_k(-1) \in K^A(E)\,. \tag{7}$$

Thus

$$\begin{aligned}&(1+y)\cdot\mathrm{mC}_y^A(B_I\subset E) = (1+y)\cdot i_{I*}\,\mathrm{mC}_y^A(id_{B_I}) =\\ &= i_{I*}i_I^*\left(\left(\prod_{k\notin I} 1 + y g^* L_k(-1)\right)\cdot g^*\,\mathrm{mC}_y^A(id_C)\cdot\lambda_y(g^*K^*(-1))\right)\\ &= i_{I*}(1)\cdot\left(\prod_{k\notin I} 1 + y g^* L_k(-1)\right)\cdot g^*\,\mathrm{mC}_y^A(id_C)\cdot\lambda_y(g^*K^*(-1))\\ &= \left(\prod_{k\in I}(1 - g^*L_k(-1))\prod_{k\notin I}(1 + y g^* L_k(-1))\right)\cdot g^*\,\mathrm{mC}_y^A(id_C)\cdot\lambda_y(g^*K^*(-1))\,.\end{aligned}$$

The second equality follows from $g_I = i_I \circ g$ and $i_I^*\mathcal{O}_{E/C}(1) = \mathcal{O}_{B_I/C}(1)$, the fourth from equation (7). By additivity of the motivic Chern class we obtain

$$(1+y)\cdot \mathrm{mC}_y^A(E\setminus B\subset E) = (1+y)\cdot \sum_I (-1)^{|I|}\,\mathrm{mC}_y^A(B_I\subset E) =$$
$$= \left(\prod_{k=1}^r (1+yg^*L_k(-1)) - (1-g^*L_k(-1))\right)\cdot g^*\,\mathrm{mC}_y^A(id_C)\cdot\lambda_y(g^*K^*(-1))$$
$$= (1+y)^r\left(\prod_{k=1}^r g^*L_k(-1)\right)\cdot g^*\,\mathrm{mC}_y^A(id_C)\cdot\lambda_y(g^*K(-1))\in K^A(E)[y]\,.$$

Therefore

$$g_*\left(\mathcal{O}_{E/C}(s)\cdot \mathrm{mC}_y^A(E\setminus B\subset E)\right) =$$
$$= \mathrm{mC}_y^A(id_C)\cdot(1+y)^{r-1}\cdot\prod_{k=1}^r L_k\cdot\left(\sum_{k=0}^{\mathrm{rk}\,V-r}\Lambda^kK\cdot g_*\mathcal{O}_{E/C}(s-r-k)\right)\,.$$

So it is enough to prove that for $1\le s\le r-1$ and $0\le k\le \mathrm{rk}\,V-r$ we have

$$g_*(\mathcal{O}_{E/C}(s-r-k)) = 0\,. \qquad (8)$$

The above inequalities imply

$$1-\mathrm{rk}\,V\le s-r-k\le -1\,.$$

So equation (8) is a direct consequence of proposition 1.10. ☐

LEMMA 3.15. *Consider the situation as in proposition 3.10. Suppose that $C\subset\bigcap_{k=1}^r\partial Y_k$ and $C\cap\partial Y_k=\varnothing$ for $k>r$. Then*

$$b_*\left(\mathcal{O}_Z(-sE)\cdot\mathrm{mC}_y^A(Z^o\subset Z)\right) = \mathrm{mC}_y^A(Y^o\subset Y)$$

for $0\le s\le r-1$.

PROOF. For $s=0$ the proposition is trivial. It is enough to prove that for $s\in\{1,2,...,r-1\}$ the pushforward of difference between

$$\mathcal{O}_Z((-s+1)E)\cdot\mathrm{mC}_y^A(Z^o\subset Z) \qquad (9)$$

and

$$\mathcal{O}_Z(-sE)\cdot\mathrm{mC}_y^A(Z^o\subset Z) \qquad (10)$$

vanishes. Note that $L:=i^*\mathcal{O}_Z(-E)$ is the relative $\mathcal{O}_{E/C}(1)$ for the projective bundle $b_{|E}\colon E\to C$. By lemma 1.20 for $D=E$ and

$$\alpha = \mathcal{O}_Z((-s+1)E)\cdot\mathrm{mC}_y^A(Z^o\subset Z)$$

we obtain that the difference (9)–(10) is equal to

$$i_*\mathcal{O}_E\cdot\mathcal{O}_Z((-s+1)E)\cdot\mathrm{mC}_y^A(Z^o\subset Z) = i_*i^*\left(\mathcal{O}_Z((-s+1)E)\cdot\mathrm{mC}_y^A(Z^o\subset Z)\right)$$
$$= i_*\left(L^{s-1}\cdot i^*\,\mathrm{mC}_y^A(Z^o\subset Z)\right)\,.$$

Moreover

$$i^*\,\mathrm{mC}_y^A(Z^o\subset Z) = (1+y)L\cdot\mathrm{mC}_y^A\left(E\setminus\bigcup_{k=0}^m\partial Z_k\right) = (1+y)L\cdot\mathrm{mC}_y^A\left(E\setminus\bigcup_{k=0}^r\partial Z_k\right)\,.$$

The first equality follows from proposition 1.83 and the second from the fact that $C \cap \partial Y_k = \varnothing$ for $k > r$. Thus, the pushforward b_* of the difference (9)–(10) is equal to

$$(1+y) \cdot i'_* g_* \left(L^s \cdot \mathrm{mC}_y^A \left(E \setminus \bigcup_{k=0}^{r} \partial Z_k \subset E \right) \right)$$

The conclusion follows from lemma 3.14 for $V = \nu(C \subset Y)$ and $V_i = \nu(C \subset \partial Y_i)$ for $i \in \{1, ..., r\}$. □

Example 3.16 (Evidence by calculus). Let

$$Y = \mathbb{A}^r , \qquad \partial Y = \bigcup_{k=1}^{r} \mathbb{A}^{r-1} , \qquad C = \{0\} .$$

We consider the natural action of the torus $A = (\mathbb{C}^*)^r$ on $\mathbb{A}^r$. Then $E = \mathbb{P}^{r-1}$ and $\partial Z \cap E$ is the sum of coordinate hyperspaces. We want to compute

$$g_* \left(\mathcal{O}_{E/C}(s) \cdot \mathrm{mC}_y^A \left(E \setminus \partial Z \subset E \right) \right) = g_* \left(\mathcal{O}_{\mathbb{P}^{r-1}}(s) \cdot \mathrm{mC}_y^A \left(\mathbb{P}^{r-1} \setminus \partial Z \subset \mathbb{P}^{r-1} \right) \right)$$

By Lefschetz-Riemann-Roch theorem (theorem 1.24) it is enough to compute restrictions of the above class to the fixed point set $(\mathbb{P}^{r-1})^A$. It can be done using the fundamental calculation of [**FRW21**, Section 2.7] and the product property of mC_y^A classes [**AMSS19**, Theorem 4.2 (3)]. We obtain that the above pushforward is equal to

$$\sum_{k=1}^{r} t_k^{-s} \prod_{l \neq k} \frac{(1+y)t_k/t_l}{1 - t_k/t_l} = (1+y)^{r-1} \prod_{l=1}^{r} t_l^{-1} \cdot \sum_{k=1}^{r} \frac{t_k^{r-s}}{\prod_{l \neq k}(1 - t_k/t_l)}$$

Due to Lefschetz-Riemann-Roch formula we have

$$\sum_{k=1}^{r} \frac{t_k^{r-s}}{\prod_{\ell \neq k}(1 - t_k/t_\ell)} = \chi(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(s-r))$$

which is zero for $s \in \{1, 2, \ldots, r-1\}$.

The working assumption of lemma 3.15 that C lies entirely in the intersection of components of ∂Y may be removed.

Lemma 3.17. *Consider the situation as in proposition 3.10. Suppose that $C \subset \bigcap_{k=1}^{r} \partial Y_k$ and C is not contained in ∂Y_k for $k > r$. Then*

$$b_* \left(\mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^o \subset Z) \right) = \mathrm{mC}_y^A(Y^o \subset Y)$$

for $0 \leq s \leq r-1$.

Proof. The subvariety C is contained in $\bigcap_{k=1}^{r} \partial Y_k$. There exists some l (possibly $l = m$) such that

$$\begin{cases} C \cap \partial Y_k = \varnothing \text{ for } k > l \,, \\ C \cap \partial Y_l \neq \varnothing \,. \end{cases}$$

We need to prove the lemma for $l \in \{r, r+1, ..., m\}$. We proceed by induction on the difference $l - r$. In the case $l = r$ the proposition simplifies to lemma 3.15. This allows to start induction.

Fix a value of $l-r$ and suppose that the lemma holds for all smaller values. The divisor ∂Z_l is the proper transform of ∂Y_l. It follows that (cf. [**Vak18**, Paragraph 22.2.6])

$$b' = b_{|\partial Z_l} \colon \partial Z_l \to \partial Y_l$$

is the blow-up of ∂Y_l in the center

$$C' = C \times_Y \partial Y_l\,.$$

The divisor ∂Y_l meets C normally, thus

$$C \cap \partial Y_l = C \times_Y \partial Y_l = C'\,.$$

The exceptional divisor of b' is $E' = E \times_Z \partial Z_l$ (cf. [**Vak18**, Paragraph 22.2.6]). Thus, $E' = E \cap \partial Z_l$ since E meets ∂Z_l transversely. The center C' of the blow up b' is entirely contained in the intersection of r divisors in ∂Y_l

$$C' \subset \bigcap_{k=1}^{r} (\partial Y_l \cap \partial Y_k)\,.$$

Moreover, for $k > l$

$$C' \cap (\partial Y_l \cap \partial Y_k) = \varnothing\,.$$

It follows that we can use the inductive assumption for the blow up b' of the pair $(\partial Y_l, \bigcup_{k\neq l} \partial Y_l \cap \partial Y_k)$ and the blow up b of the pair $(Y, \bigcup_{k\neq l} \partial Y_k)$. Let us recall that

$$Y^o = Y \setminus \bigcup_{k=0}^{m} \partial Y_k\,, \qquad Z^o = Z \setminus \left(E \cup \bigcup_{k=0}^{m} \partial Z_k \right)$$

and introduce the following notation

$$\partial Y_l^o = \partial Y_l \setminus \bigcup_{k\neq l}^{m} (\partial Y_k \cap \partial Y_l)\ , \quad \partial Z_l^o = \partial Z_l \setminus \left((E \cap \partial Z_l) \cup \bigcup_{k\neq l}^{m} (\partial Z_k \cap \partial Z_l) \right)\,.$$

Consider the disjoint unions

$$Y^\# := Y^o \cup \partial Y_l^o\,, \qquad Z^\# := Z^o \cup \partial Z_l^o\,.$$

Denote by $j \colon \partial Y_l \to Y$ and $\tilde{j} \colon \partial Z_l \to Z$ the inclusions. By the inductive assumption we have

$$b_*\big(\mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^\# \subset Z)\big) = \mathrm{mC}_y^A(Y^\# \subset Y)$$

and

$$b'_*\big(\mathcal{O}_Z(-sE') \cdot \mathrm{mC}_y^A(\partial Z_l^o \subset \partial Z_l)\big) = \mathrm{mC}_y^A(\partial Y_l^o \subset \partial Y_l)\,.$$

Therefore, by additivity of the motivic Chern class

$$\begin{aligned}
&b_*\big(\mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^o \subset Z)\big) = \\
&= b_*\big(\mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^\# \subset Z)\big) - b_*\big(\mathcal{O}_Z(-sE) \cdot \tilde{j}_* \mathrm{mC}_y^A(\partial Z_l^o \subset \partial Z_l)\big) \\
&= b_*\big(\mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^\# \subset Z)\big) - b_*\tilde{j}_*\big(\mathcal{O}_Z(-sE') \cdot \mathrm{mC}_y^A(\partial Z_l^o \subset \partial Z_l)\big) \\
&= b_*\big(\mathcal{O}_Z(-sE) \cdot \mathrm{mC}_y^A(Z^\# \subset Z)\big) - j_*b'_*\big(\mathcal{O}_Z(-sE') \cdot \mathrm{mC}_y^A(\partial Z_l^o \subset \partial Z_l)\big) \\
&= \mathrm{mC}_y^A(Y^\# \subset Y) - j_*\big(\mathrm{mC}_y^A(\partial Y_l^o \subset \partial Y_l)\big) \\
&= \mathrm{mC}_y^A(Y^o \subset Y)
\end{aligned}$$

□

This ends the proof of proposition 3.10.

CHAPTER 4

Twisted motivic Chern class as stable envelope

4.1. Statement of result

In this chapter our aim is a generalization of results of chapter 2 to the case of an arbitrary slope. Let A be an algebraic torus. Let M be a smooth, projective A-variety. Suppose that the fixed point set M^A is finite. Consider the induced action of the product torus $\mathbb{T} = A \times \mathbb{C}^*$ on the cotangent variety $X = T^*M$ defined as in example 1.61. Denote by π the projection

$$\pi \colon T^*M \to M.$$

Let $\sigma \colon \mathbb{C}^* \to A$ be a good one parameter subgroup (see definition 1.62). For a fixed point $\mathbf{e} \in M^A$ let

$$i_{\mathbf{e}} \colon \overline{M_{\mathbf{e}}^+} \hookrightarrow M$$

be an inclusion of the closure of the BB-cell. Let

$$\partial\overline{M_{\mathbf{e}}^+} := \overline{M_{\mathbf{e}}^+} \setminus M_{\mathbf{e}}^+$$

denote the boundary of the BB-cell. Consider a slope $s \in Pic(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. In section 4.2 we will associate with it a $\mathbb{Q}$-Cartier divisor $\Delta_{\mathbf{e},s}$ on $\overline{M_{\mathbf{e}}^+}$ with support contained in the boundary $\partial\overline{M_{\mathbf{e}}^+}$. We prove that after a suitable normalization the twisted motivic Chern class of BB-cell satisfies all but one of the axioms of the stable envelope for the slope s.

Definition 4.1. Suppose that Y is a $\mathbb{T}$-variety, such that the factor $\mathbb{C}^*$ acts trivially. Let ρ be a map

$$\rho \colon K^A(Y)[y] \to K^{\mathbb{T}}(Y)$$

given by $\rho(y) = -\mathfrak{h}$.

Theorem 4.2. *Let $\mathbf{e} \in M^A$ be a fixed point. The class*

$$\mathfrak{h}^{-\dim(M_{\mathbf{e}}^+)}\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+};\partial\overline{M_{\mathbf{e}}^+};\Delta_{\mathbf{e},s})) \in K^{\mathbb{T}}(X)$$

satisfies the normalization axiom and the Newton inclusion property of the stable envelope $Stab^s(\mathbf{e})$. Moreover, it satisfies the distinguished point axiom.

Corollary 4.3. *Suppose that the slope s is small and antiample. Then the divisor $-\Delta_{\mathbf{e},s}$ is small and effective. Due to proposition 3.7 the class*

$$\mathfrak{h}^{-\dim(M_{\mathbf{e}}^+)}\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+};\partial\overline{M_{\mathbf{e}}^+};\Delta_{\mathbf{e},s})) = \mathfrak{h}^{-\dim(M_{\mathbf{e}}^+)}\pi^*\rho(\mathrm{mC}_y^A(M_{\mathbf{e}}^+ \to M))$$

satisfies all but one axioms of the stable envelope $Stab^s(\mathbf{e})$.

The above theorem is a direct consequence of propositions 4.10, 4.11 and 4.17.

Remark 4.4. In this chapter we do not assume any regularity conditions on the BB-decomposition of M. As noted in remark 1.71 the support axiom cannot be stated on this level of generality.

4.2. Boundary divisor

We consider the situation described in the previous section. Let $\mathcal{L} \in Pic(M)$ be a line bundle. We will show that a slope

$$s = \mathcal{L}^{1/n} \in Pic(M) \otimes \mathbb{Q}$$

determines a $\mathbb{Q}$-Cartier divisor $\Delta_{\mathbf{e},s}$ on $\overline{M^+_{\mathbf{e}}}$. By [**Bri15**, Lemma 2.14] we can assume that $\mathcal{L}$ admits an A-linearisation and we fix an A-linearisation.

Definition 4.5. Let $\mathcal{L} \in Pic^A(M)$ be a linearised line bundle. We say that a meromorphic section $\mathbf{v} \in i^*_{\mathbf{e}}\mathcal{L}(U)$ of the bundle $i^*_{\mathbf{e}}\mathcal{L}$ is good if

- the section $\mathbf{v}$ is an eigenvector of the torus A,
- the section $\mathbf{v}$ does not vanish nor has a pole at the center of the cell.

For a slope $s = \mathcal{L}^{1/n}$ any good section of $\mathcal{L}$ defines the divisor

$$\Delta_{\mathbf{e},s} = \frac{1}{n}\operatorname{div}(\mathbf{v}).$$

Straight from the definition we obtain the following properties of divisor $\Delta_{\mathbf{e},s}$.

Proposition 4.6. *Let $\mathbf{v}$ be a good section of $i^*_{\mathbf{e}}\mathcal{L}$. Let $\Delta_{\mathbf{e},s} = \frac{1}{n}div(\mathbf{v})$. Then*

(1) Divisor $\Delta_{\mathbf{e},s}$ is $\mathbb{Q}$-Cartier.
(2) Divisor $\Delta_{\mathbf{e},s}$ is A-invariant.
(3) Support of $\Delta_{\mathbf{e},s}$ is contained in the boundary $\partial\overline{M^+_{\mathbf{e}}}$.

In this section we prove that good sections exist and the divisor $\Delta_{\mathbf{e},s}$ associated to the slope $s = \mathcal{L}^{1/n}$ is unique.

Proposition 4.7. *There exists a good section $\mathbf{v}$ of $i^*_{\mathbf{e}}\mathcal{L}$.*

Proof. First, assume that $\mathcal{L}$ is generated by global sections. The torus A acts on the finitely dimensional vector space of global sections $H^0(\overline{M^+_{\mathbf{e}}}; i^*\mathcal{L})$. Since the eigenvectors span that space we can find a section $\mathbf{v}$ such that

- $\mathbf{v} \in H^0(\overline{M^+_{\mathbf{e}}}; i^*\mathcal{L})$ is an eigenvector of A,
- $\mathbf{v}(\mathbf{e}) \neq 0$.

In particular, it follows that all very ample line bundles admit a good section. If $\mathcal{L}$ is not globally generated we can tensor it with a suitable power of an equivariant ample bundle $\mathcal{O}(1)$ and argue as before. If $\mathbf{v}_1$ is a good section for $\mathcal{L}(n)$ and $\mathbf{v}_2$ is a good section of $\mathcal{O}(n)$ then $\mathbf{v}_1/\mathbf{v}_2$ is a good section of $\mathcal{L}$. □

Remark 4.8. Due to [**Bri15**, Proposition 2.10] any linearization of $i^*_{\mathbf{e}}\mathcal{L}$ differs from the natural linearization of $\mathcal{O}(n\Delta_{\mathbf{e},s})$ by a twist. Therefore, for a line bundle $\mathcal{L}$ considered with a fixed linearization and $\mathcal{O}(n\Delta_{\mathbf{e},s})$ with the natural linearization we have

$$i^*_{\mathbf{e}}\mathcal{L} = t^{\mathrm{w}_{\mathbf{e}}(\mathcal{L})} \cdot \mathcal{O}(n\Delta_{\mathbf{e},s}) \in K^A(\overline{M^+_{\mathbf{e}}})\,.$$

Proposition 4.9. *Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two good meromorphic sections of $i^*_{\mathbf{e}}\mathcal{L}$ on $\overline{M^+_{\mathbf{e}}}$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are proportional.*

PROOF. The sections $\mathbf{v}_i$ are defined on an A-equivariant open neighbourhood $U \subset \overline{M_{\mathbf{e}}^+}$ containing $\mathbf{e}$. Let $\chi_i \colon A \to \mathbb{C}^*$ be the corresponding characters, i.e.

$$\mathbf{t} \cdot \mathbf{v}_i = \chi_i(\mathbf{t})\mathbf{v}$$

for $\mathbf{t} \in A$ and $i \in \{1, 2\}$. The quotient $\mathbf{v}_1/\mathbf{v}_2$ defines a rational function

$$\theta := \frac{\mathbf{v}_1}{\mathbf{v}_2} \colon U \to \mathbb{C}\,.$$

The map θ is A-equivariant if we consider the action of the torus A on $\mathbb{C}$ given by the character $\frac{\chi_1}{\chi_2}$. Moreover, θ does not have zero nor pole at $\mathbf{e}$. Since $\mathbf{e}$ is a fixed point, $\theta(\mathbf{e}) \neq 0$ is fixed as well. Thus, the action of A on $\mathbb{C}$ is trivial. It follows that characters χ_1 and χ_2 coincide and the map θ is constant on the orbits of A. The map θ is defined at $\mathbf{e}$ so it is defined and constant on the whole BB-cell $M_{\mathbf{e}}^+$. It follows that θ is also defined and constant on the closure $\overline{M_{\mathbf{e}}^+}$ of the BB-cell. Thus, $\theta = \mathbf{v}_1/\mathbf{v}_2$ is constant. □

4.3. Normalization axiom

We use notation from section 4.1.

PROPOSITION 4.10. *Consider a fixed point* $\mathbf{e} \in M^A$. *The class*

$$\mathfrak{h}^{-\dim(M_{\mathbf{e}}^+)}\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}; \partial\overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s})) \in K^{\mathbb{T}}(X)$$

satisfies the normalization axiom of the stable envelope $Stab^s(\mathbf{e})$. *Namely*

$$\mathfrak{h}^{-\dim(M_{\mathbf{e}}^+)}\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}; \partial\overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s}))_{|\mathbf{e}} = (-1)^{\dim(M_{\mathbf{e}}^+)}\frac{eu(T_{\mathbf{e}}^-X)}{\det(T_{\mathbf{e}}^+M)}$$

PROOF. The point $\mathbf{e}$ lies in the interior of the BB-cell $M_{\mathbf{e}}^+$. Therefore, the standard and the twisted motivic Chern class coincide at $\mathbf{e}$ (due to proposition 3.9 (1))

$$i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}; \partial\overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s})_{|\mathbf{e}} = \mathrm{mC}_y^A(M_{\mathbf{e}}^+ \subset M)_{|\mathbf{e}}.$$

The proposition follows from proposition 2.5. □

4.4. Newton inclusion property

We use notation from section 4.1.

PROPOSITION 4.11. *Consider a fixed point* $\mathbf{e} \in M^A$. *The class*

$$\mathfrak{h}^{-\dim(M_{\mathbf{e}}^+)}\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}; \partial\overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s}))$$

satisfies the Newton polytope axiom of the stable envelope $Stab^s(\mathbf{e})$. *For any fixed point* $\mathbf{e}' < \mathbf{e}$ *we have*

$$\mathcal{N}^{\mathrm{A}}\left(\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}; \partial\overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s})_{|\mathbf{e}'})\right) \subset \mathcal{N}^{\mathrm{A}}\left(eu(T_{\mathbf{e}'}^-X)\right) - \mathrm{w}_{\mathbf{e}'}(\det T_{\mathbf{e}'}^+M) + \mathrm{w}_{\mathbf{e}'}(s) - \mathrm{w}_{\mathbf{e}}(s)$$

According to lemma 2.8 and remark 4.8, we only need to prove that for a fixed point $\mathbf{e}' < \mathbf{e}$ there is an inclusion:

$$\mathcal{N}^{\mathrm{A}}\left(i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}; \partial\overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s})_{|\mathbf{e}'}\right) - \mathrm{w}_{\mathbf{e}'}(\Delta_{\mathbf{e},s}) \subset \mathcal{N}^{\mathrm{A}}\left(eu\left(T_{\mathbf{e}'}M\right)\right)\,. \qquad (11)$$

Consider an $A-$equivariant SNC resolution of singularities

$$f \colon (Y, \partial Y) \to (\overline{M_{\mathbf{e}}^+}; \partial\overline{M_{\mathbf{e}}^+})\,.$$

Set $\Delta = \Delta_{\mathbf{e},s}$ and $D = f^*\Delta$. For a fixed point set component

$$F \subset Y^A \cap f^{-1}(\mathbf{e}')$$

denote by f_F the map to a point $f_{|F}\colon F \to \mathbf{e}'$. The Lefschetz-Riemann-Roch formula 1.24 implies that

$$\begin{aligned}\frac{i_{\mathbf{e}*}\,\mathrm{mC}_y^A\left(\overline{M^+_{\mathbf{e}}},\partial\overline{M^+_{\mathbf{e}}};\Delta\right)_{|\mathbf{e}'}}{eu(T_{\mathbf{e}'}M)} &= \frac{f_*i_{\mathbf{e}*}\,\mathrm{mC}_y^A\left(Y,\partial Y;D\right)_{|\mathbf{e}'}}{eu(T_{\mathbf{e}'}M)}\\ &= \sum_{F\subset f^{-1}(\mathbf{e}')\cap Y^A} f_{F*}\left(\frac{\mathrm{mC}_y^A\left(Y,\partial Y;D\right)_{|F}}{eu\left(\nu(F\subset Y)\right)}\right).\end{aligned}$$

This formula can be rewritten as

(12)
$$\frac{t^{\mathrm{w}_{\mathbf{e}'}(-\Delta)}\,i_{\mathbf{e}*}\,\mathrm{mC}_y^A\left(\overline{M^+_{\mathbf{e}}},\partial\overline{M^+_{\mathbf{e}}};\Delta)\right)_{|\mathbf{e}'}}{eu(T_{\mathbf{e}'}M)} = \sum_{F\subset f^{-1}(\mathbf{e}')\cap Y^A} f_{F*}\left(\frac{t^{\mathrm{w}_F(-D)}\,\mathrm{mC}_y^A\left(Y,\partial Y;D\right)_{|F}}{eu\left(\nu(F\subset Y)\right)}\right).$$

REMARK 4.12. In the above formula we use the extended K-theory ring $\widetilde{K}(\mathbf{e}')$ (see section 1.2.3).

Proposition 1.52 implies that to prove the inclusion (11) it is enough to show that if a one parameter subgroup $\sigma'\colon \mathbb{C}^* \to A$ is general enough then the limit $\lim_{\sigma'}$ of a single summand occurring in the equation (12) exists. Using commutation of the limit map with push-forwards (see proposition 1.43) we only need to prove the following lemma:

LEMMA 4.13. *Let Y be a smooth projective A-variety with a chosen SNC divisor ∂Y. Let D be a $\mathbb{Q}$-Cartier A-invariant SNC divisor on Y with support contained in ∂Y. Let $F \subset Y^A$ be a component of the fixed point set. Then the limit*

$$\lim_{\sigma'} \frac{t^{-\mathrm{w}_F(D)}\,\mathrm{mC}_y^A\left(Y,\partial Y;D\right)_{|F}}{eu\left(\nu(F\subset Y)\right)} \in K(F)[y]$$

exists for a general one parameter subgroup $\sigma'\colon \mathbb{C}^* \to A$.

Proof of Lemma 4.13. Let $Y^o = Y\setminus \partial Y$ and $\partial Y = \bigcup_{i=1}^m D_i$. For a subset of indices $I\subset \{1,...,m\}$ consider the bundle:

$$L_I = \prod_{j\in I} \mathcal{O}_Y(D_j)_{|F}\,.$$

The weight

$$\mathrm{w}_F(\mathcal{O}_Y(\lceil D\rceil)_{|F}) - \mathrm{w}_F(D)$$

lies in the convex hull of the weights $\mathrm{w}_F(L_I)$. By propositions 1.54 and 1.41 it is enough to show that the limits

(13)
$$\lim_{\sigma'} \frac{L_I\cdot \mathrm{mC}_y^A(Y^o\subset Y)_{|F}}{eu\left(\nu(F\subset Y)\right)} = \lim_{\sigma'} \frac{\mathrm{mC}_y^A\left(Y^o\subset Y\right)_{|F}}{eu\left(\nu(F\subset Y)\right)}\prod_{j\in I}\mathcal{O}_Y(D_j)_{|F}$$

exist. The following simple observation allows to remove some factors from the above expression.

LEMMA 4.14. *Consider a component $D_j \subset Y$. Let $F \subset Y^A$ be a component of the fixed point set. Suppose that F is not contained in D_j. Then the weight of the bundle $\mathcal{O}_Y(D_j)$ at F is equal to zero, i.e.*

$$\mathrm{w}_F(\mathcal{O}_Y(D_j)) = 0\,.$$

PROOF. We have

$$F \setminus D_j \neq \varnothing\,.$$

Hence, in a point of F not belonging to D_j the weight of $\mathcal{O}_Y(D_j)$ is trivial. The weight does not depend on the choice of a point in F. □

Thus, when the variety F is not contained in D_j we may omit the bundle $\mathcal{O}_Y(D_j)$ in the expression (13). Existence of the considered limit follows from the following lemma.

LEMMA 4.15. *Consider the situation as in lemma 4.13. Suppose that F is a component of the fixed point set Y^A. Let $\sigma'\colon \mathbb{C}^* \to A$ be a good one parameter subgroup for Y (i.e. $Y^A = Y^{\mathbb{C}^*}$). Then the limit*

$$\lim_{\sigma'} \frac{\mathrm{mC}^A_y\,(Y^o \subset Y)_{|F}}{eu(\nu(F \subset Y))} \prod_{j\in I} \mathcal{O}_Y(D_j)_{|F}$$

exists. Moreover, if all the weights of the line bundles $\mathcal{O}_Y(D_j)_{|F}$ are negative then the limit is equal to

$$(1+y)^{|I|}\,\mathrm{mC}^A_y\left((D^o_I)^{\sigma'+}_F \to F\right)\,.$$

Otherwise the limit is equal to 0. Here

$$(D^o_I)^{\sigma'+}_F = \{x \in D^o_I \;:\; \lim_{t\to 0}\sigma'(t)\cdot x \in F\,\}\,.$$

PROOF. Corollary 1.87 implies that

$$\lim_{\sigma'} \frac{\mathrm{mC}^A_y\,(Y^o \subset Y)_{|F}}{eu(\nu(F \subset Y))} \prod_{j\in I} \mathcal{O}_Y(D_j)_{|F} = \lim_{\sigma'} \frac{\mathrm{mC}^A_y\,(D^o_I \subset D_I)_{|F}}{eu(\nu(F \subset D_I))}(1+y)^{|I|} \prod_{j\in I} \frac{1}{1-\mathcal{O}_Y(-D_j)_{|F}}\,.$$

Theorem 1.81 implies that the limit

$$\lim_{\sigma'} \frac{\mathrm{mC}^A_y\,(D^o_I \subset D_I)_{|F}}{eu(\nu(F \subset D_I))}$$

exists and is equal to

$$\mathrm{mC}^A_y\left((D^o_I)^{\sigma'+}_F \to F\right)\,.$$

Moreover

$$\lim_{\sigma'} \frac{1}{1-\mathcal{O}_Y(-D_j)_{|F}} = \begin{cases} 0 & \text{if the weight of } \mathcal{O}_Y(D_j)_{|F} \text{ is positive,} \\ 1 & \text{if the weight of } \mathcal{O}_Y(D_j)_{|F} \text{ is negative.} \end{cases}$$

Multiplying these limits we arrive at the desired equality. □

REMARK 4.16. The weight of $\mathcal{O}_Y(D_j)_{|F}$ is nontrivial because we assumed that $Y^{\mathbb{C}^*} = Y^A$ and F is a fixed point set component.

4.5. Distinguished point

We use notation from section 4.1.

PROPOSITION 4.17. *For a pair of fixed points* $\mathbf{e}', \mathbf{e} \in M^A$ *such that* $\mathbf{e} > \mathbf{e}'$, *we have* $-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M) + \mathrm{w}_{\mathbf{e}'}(s) - \mathrm{w}_{\mathbf{e}}(s) \notin \mathcal{N}^{\mathrm{A}}\left(\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}^A_y(\overline{M^+_{\mathbf{e}}};\partial\overline{M^+_{\mathbf{e}}};\Delta_{\mathbf{e},s}))_{|\mathbf{e}'}\right)$.

PROOF. It is enough to prove that

$$-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M) \notin \mathcal{N}^{\mathrm{A}}\left(i_{\mathbf{e}*}\,\mathrm{mC}^A_y(\overline{M^+_{\mathbf{e}}};\partial\overline{M^+_{\mathbf{e}}};\Delta_{\mathbf{e},s})_{|\mathbf{e}'}\right) - \mathrm{w}_{\mathbf{e}'}(\Delta_{\mathbf{e},s})\,.$$

Consider an $A-$equivariant SNC resolution of singularities

$$f\colon (Y,\partial Y) \to (\overline{M^+_{\mathbf{e}}};\partial\overline{M^+_{\mathbf{e}}}).$$

Let $\sigma'\colon \mathbb{C}^* \to A$ be a one parameter subgroup such that σ' is good for Y (i.e. $Y^A = Y^{\mathbb{C}^*}$) and lies in the same weight chamber as the one parameter subgroup σ (with respect to the variety M). Let

$$\sigma'^*\colon K^A(M) \to K^{\mathbb{C}^*}(M)\,, \qquad \pi_{\sigma'}\colon \mathrm{Hom}(A,\mathbb{C}^*)\otimes\mathbb{R} \to \mathrm{Hom}(\mathbb{C}^*,\mathbb{C}^*)\otimes\mathbb{R}\,,$$

be maps induced be the one parameter subgroup σ. The point $\pi_{\sigma'}(-\mathrm{w}_{\mathbf{e}'}(\det T^+_{\mathbf{e}'}M))$ is the lowest term of the line segment $N^{\sigma'}\left(\sigma'^*eu(T^-_{\mathbf{e}'}X)\right)$. Therefore, it is enough to prove that the limit

$$\lim_{\sigma'} \frac{t^{\mathrm{w}_{\mathbf{e}'}(-\Delta_{\mathbf{e},s})}\,i_{\mathbf{e}*}\,\mathrm{mC}^A_y\left(\overline{M^+_{\mathbf{e}}},\partial\overline{M^+_{\mathbf{e}}};\Delta_{\mathbf{e},s})\right)_{|\mathbf{e}'}}{eu(T_{\mathbf{e}'}M)}$$

is equal to zero. We use notations from the previous section, in which we proved that this limit exists. It is enough to show that the limit of a single summand of expression (12) vanishes. Coefficients of the $\mathbb{Q}$-divisor

$$\lceil D\rceil - D$$

are rational numbers from the interval $[0,1)$. Therefore, the fractional weight

$$\mathrm{w}_F(\mathcal{O}_Y(\lceil D\rceil)_{|F}) - \mathrm{w}_F(D)$$

can be expressed as a convex combination of the weights $\mathrm{w}_F(L_I)$ with a nonzero coefficient at $\mathrm{w}_F(L_\varnothing)$. The previous section implies that all the limits

$$\lim_{\sigma'} \frac{L_I\cdot \mathrm{mC}^A_y(Y^o\subset Y)_{|F}}{eu\,(\nu(F\subset Y))}$$

exist. Moreover

$$\lim_{\sigma'} \frac{L_\varnothing\cdot \mathrm{mC}^A_y(Y^o\subset Y)_{|F}}{eu\,(\nu(F\subset Y))} = \lim_{\sigma'} \frac{\mathrm{mC}^A_y(Y^o\subset Y)_{|F}}{eu\,(\nu(F\subset Y))} =$$
$$= \mathrm{mC}_y(Y^o\cap Y^+_F \to F) = \mathrm{mC}_y(\varnothing\to F) = 0\,.$$

Where the second equality follows from theorem 1.81 and the third from $Y^o = Y^+_{\mathbf{e}}$. Proposition 1.53 ends the proof. □

REMARK 4.18. If the one parameter subgroup σ is good for Y, then we may take $\sigma' = \sigma$.

CHAPTER 5

Support axiom

In the previous sections we proved that the (twisted) motivic Chern class of BB-cell satisfies all but one of the axioms of the stable envelope for an arbitrary projective variety M. The remaining axiom is of different nature. To even state it we need some regularity condition on the BB-decomposition of M (see remark 1.71).

In this chapter we prove that for an interesting class of examples the twisted motivic Chern class satisfies also the support axiom. First, we give an equivalent statement of the axiom in terms of divisibility (see proposition 5.2). Then, we present a condition on the BB-stratification on M which is sufficient to prove that the twisted motivic Chern class satisfies the support axiom. Lastly, we prove that homogenous varieties satisfy this condition. The main results of this chapter are theorems 5.12 and 5.15.

5.1. Equivalent statement of the support axiom

Let A be an algebraic torus and $\mathbb{T} = A \times \mathbb{C}^*$. Let M be a projective smooth A-variety and $X = T^*M$ the cotangent variety with the induced $\mathbb{T}$-action (see example 1.61). Suppose that the fixed point set M^A is finite. Let $\sigma\colon \mathbb{C}^* \to A$ be a one parameter subgroup such that the pair (M, σ) is admisible (definition 1.64), i.e. that the set

$$\bigcup_{\mathbf{e}\in M^A} X^+_{\mathbf{e}} = \bigcup_{\mathbf{e}\in M^A} \nu^*\left(M^+_{\mathbf{e}} \subset M\right)$$

is closed in X. The support axiom of the stable envelope states that

$$\operatorname{supp}(stab(\mathbf{e})) \subset \bigcup_{\mathbf{e}'\le\mathbf{e}} X^+_{\mathbf{e}'}\,.$$

Remark 5.1. The support axiom of the stable envelope does not depend on a slope. Therefore, in this chapter we omit slope in the notation.

Proposition 5.2 (cf. [**RTV15**, Remark after theorem 3.1], [**RTV19**, Lemma 5.2-4]). *Consider the above situation. Let $\mathbf{e} \in M^A$ be a fixed point and $a \in K^{\mathbb{T}}(X)$ a K-theory class. The following conditions are equivalent*

1) *The element a satisfies the support axiom*

$$\operatorname{supp}(a) \subset \bigcup_{\mathbf{e}'\le\mathbf{e}} X^+_{\mathbf{e}'}\,.$$

2) *For any fixed point $\mathbf{e}' \in M^A$ the restriction $a_{|\mathbf{e}'}$ may be nonzero only if $\mathbf{e}' \le \mathbf{e}$. In that case $a_{|\mathbf{e}'}$ is divisible by the class $\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}'}M)^*)$, i.e.*

$$\begin{cases}\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}'}M)^*) \mid a_{|\mathbf{e}'} \text{ if } \mathbf{e}' \le \mathbf{e},\\ a_{|\mathbf{e}'} = 0 \text{ otherwise.}\end{cases}$$

For completeness we give a proof of the above proposition. We need some notations and technical lemmas. For a fixed point $\mathbf{e} \in M^A$ let

$$V_{\mathbf{e}} = T^*M_{|M_{\mathbf{e}}^+}\,, \qquad\qquad U_{\mathbf{e}} = T^*M_{|M_{\mathbf{e}}^+} \setminus X_{\mathbf{e}}^+\,.$$

LEMMA 5.3. *Consider the situation as in proposition 5.2. Let $\mathbf{e} \in M^A$ be a fixed point. The following are equivalent*

1) *The element a restricted to $V_{\mathbf{e}}$ vanishes $a_{|V_{\mathbf{e}}} = 0$.*
2) *The element a restricted to the point $\mathbf{e}$ vanishes $a_{|\mathbf{e}} = 0$.*

PROOF. The subvariety $V_{\mathbf{e}}$ is an affine space. It follows that the restriction to a point induces an isomorphism

$$K^{\mathbb{T}}(V_{\mathbf{e}}) \simeq K^{\mathbb{T}}(\mathbf{e})\,.$$

□

LEMMA 5.4. *Consider the situation as in proposition 5.2. Let $\mathbf{e} \in M^A$ be a fixed point. The following are equivalent*

1) *The element a restricted to $U_{\mathbf{e}}$ vanishes $a_{|U_{\mathbf{e}}} = 0$.*
2) *The restriction $a_{|\mathbf{e}}$ is divisible by the class $\lambda_{-h}((T_{\mathbf{e}}^+M)^*)$ i.e.*

$$\lambda_{-h}((T_{\mathbf{e}}^+M)^*) \mid a_{|\mathbf{e}}\,.$$

PROOF. The subvariety $X_{\mathbf{e}}^+ \subset V_{\mathbf{e}}$ is invariant and closed. Moreover, the varieties $U_{\mathbf{e}}, V_{\mathbf{e}}$ and $X_{\mathbf{e}}^+$ are smooth. The exact sequence of a closed immersion 1.14 for $X_{\mathbf{e}}^+ \subset V_{\mathbf{e}}$ is of the form

$$\begin{array}{ccccccc}
K^{\mathbb{T}}(X_{\mathbf{e}}^+) & \longrightarrow & K^{\mathbb{T}}(V_{\mathbf{e}}) & \longrightarrow & K^{\mathbb{T}}(U_{\mathbf{e}}) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \| & & \\
K^{\mathbb{T}}(\mathbf{e}) & \xrightarrow{\alpha} & K^{\mathbb{T}}(\mathbf{e}) & \longrightarrow & K^{\mathbb{T}}(U_{\mathbf{e}}) & \longrightarrow & 0
\end{array}$$

The vertical isomorphism are induced by the restriction to the point $\mathbf{e}$. The formula 1.18 implies that the map α is a multiplication by the class $eu(X_{\mathbf{e}}^+ \subset V_{\mathbf{e}})_{|\mathbf{e}}$. Therefore,

$$\begin{aligned}
a_{|U_{\mathbf{e}}} = 0 &\iff (a_{|V_{\mathbf{e}}})_{|U_{\mathbf{e}}} = 0 \iff eu(X_{\mathbf{e}}^+ \subset V_{\mathbf{e}})_{|\mathbf{e}} \mid (a_{|V_{\mathbf{e}}})_{|\mathbf{e}} \\
&\iff \lambda_{-1}(\nu_{\mathbf{e}}^*(X_{\mathbf{e}}^+ \subset V_{\mathbf{e}})) \mid a_{|\mathbf{e}} \\
&\iff \lambda_{-1}(\nu_{\mathbf{e}}(X_{\mathbf{e}}^+ \subset V_{\mathbf{e}})) \mid a_{|\mathbf{e}} \\
&\iff \lambda_{-1}(\mathbb{C}_{\mathfrak{h}} \otimes (T_{\mathbf{e}}^+M)^*) \mid a_{|\mathbf{e}} \\
&\iff \lambda_{-\mathfrak{h}}((T_{\mathbf{e}}^+M)^*) \mid a_{|\mathbf{e}}\,.
\end{aligned}$$

The equivalence

$$\lambda_{-1}(\nu_{\mathbf{e}}^*(X_{\mathbf{e}}^+ \subset V_{\mathbf{e}})) \mid a_{|\mathbf{e}} \iff \lambda_{-1}(\nu_{\mathbf{e}}(X_{\mathbf{e}}^+ \subset V_{\mathbf{e}})) \mid a_{|\mathbf{e}}$$

follows from proposition 1.17. □

LEMMA 5.5. *Let R be a domain and $R[y, y^{-1}]$ the ring of Laurent polynomials. Assume that $A(y) \in R[y, y^{-1}]$ is a monic Laurent polynomial (the coefficient corresponding to the smallest or the greatest power of y is equal to one). Let $r \in R$ be a nonzero element. Then for any polynomial $B(y) \in R[y, y^{-1}]$*

$$A(y) \mid B(y) \iff A(y) \mid rB(y)\,.$$

PROOF. It is an easy algebra exercise. □

LEMMA 5.6. *Consider the situation as in proposition 5.2. Let* $\mathbf{e} \in M^A$ *be a fixed point. Let* $\pi_{\mathbf{e}} \colon U_{\mathbf{e}} \to M^+_{\mathbf{e}}$ *be a restriction of the projection* $\pi \colon X \to M$*. The element*

$$\pi^*_{\mathbf{e}} eu(M^+_{\mathbf{e}} \subset M)$$

is not a zero divisor in $K^{\mathbb{T}}(U_{\mathbf{e}})$*.*

PROOF. The proof of lemma 5.4 implies that

$$\begin{aligned} K^{\mathbb{T}}(U_{\mathbf{e}}) &= K^{\mathbb{T}}(\mathbf{e})/\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*) \\ &= K^A(\mathbf{e})[\mathfrak{h}, \mathfrak{h}^{-1}]/\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*)\,. \end{aligned}$$

Where $K^{\mathbb{T}}(\mathbf{e})/\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*)$ denotes the ring $K^{\mathbb{T}}(\mathbf{e})$ divided by the ideal generated by $\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*)$. We need to prove that for an element $x \in K^A(\mathbf{e})[\mathfrak{h}, \mathfrak{h}^{-1}]$ divisibility

$$\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*) \mid eu(M^+_{\mathbf{e}} \subset M)_{|\mathbf{e}} \cdot x$$

implies

$$\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*) \mid x\,.$$

Note that the ring $K^A(\mathbf{e})$ is a domain and the polynomial $\lambda_{-\mathfrak{h}}((T^+_{\mathbf{e}}M)^*)$ is monic (its smallest coefficient with respect to $\mathfrak{h}$ is equal to one). Lemma 5.5 for $R = K^A(\mathbf{e})$ and $r = eu(M^+_{\mathbf{e}} \subset M)_{|\mathbf{e}}$ completes the proof. □

PROOF OF PROPOSITION 5.2. Choose a fixed point $\mathbf{e} \in M^A$. By [**BB76**, Theorem 3] the BB-decomposition of M is filtrable. We may order the fixed point set $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ in such a way that all the subsets

$$\bigcup_{i=1}^{t} M^+_{\mathbf{e}_i} \subset M$$

are open subvarieties. Moreover, we may assume that for a chosen fixed point $\mathbf{e}$ we have $\mathbf{e} = \mathbf{e}_r$ for some r and $\mathbf{e} > \mathbf{e}_s$ if and only if $s > r$. Let

$$U = \bigcup_{i=1}^{r-1} V_{\mathbf{e}_i} \cup \bigcup_{i=r}^{n} U_{\mathbf{e}_i}\,.$$

The subset U is the complement of the closed subset $\bigcup_{\mathbf{e}' \le \mathbf{e}} X^+_{\mathbf{e}}$, so it is open (cf. definition 1.64).

1) $\Rightarrow$ **2)** We know that

$$\mathrm{supp}(a) \subset X^+_{\mathbf{e}} \Rightarrow a_{|U} = 0 \Rightarrow \begin{cases} a_{|U_{\mathbf{e}_i}} = 0 \text{ for } i \ge r\,, \\ a_{|V_{\mathbf{e}_i}} = 0 \text{ for } i < r\,. \end{cases}$$

By lemmas 5.3 and 5.4 this is equivalent to condition **2)**.

2) $\Rightarrow$ **1)** By lemmas 5.3 and 5.4 we know that

$$\begin{cases} a_{|U_{\mathbf{e}_i}} = 0 \text{ for } i \ge r\,, \\ a_{|V_{\mathbf{e}_i}} = 0 \text{ for } i < r\,. \end{cases} \tag{14}$$

Our goal is to glue these equalities to obtain $a_{|U} = 0$. Let

$$\tilde{V}_t = \bigcup_{i=1}^{t} V_{\mathbf{e}_i}\,, \qquad \tilde{U}_t = U \cap \tilde{V}_t\,, \qquad X_t = \begin{cases} U_{\mathbf{e}_t} \text{ for } t \geq r \\ V_{\mathbf{e}_t} \text{ for } t < r\,. \end{cases}$$

The inclusion $V_{\mathbf{e}_t} \subset \tilde{V}_t$ has a normal bundle equal to the pullback of the bundle $\nu(M^+_{\mathbf{e}_t} \subset M)$. The inclusion $X_t \subset \tilde{U}_t$ is restriction of $V_t \subset \tilde{V}_t$ to the open subset $\tilde{U}_t$. Its normal bundle is equal to the pullback of the bundle $\nu(M^+_{\mathbf{e}_t} \subset M)$.

We want to prove that $a_{|\tilde{U}_t} = 0$ for all t. We proceed by induction. For $t = 1$ we have $\tilde{U}_1 = U_1$ and the inductive thesis follows from equations (14). Let us focus on the inductive step. The inductive assumption states that $a_{|\tilde{U}_{t-1}} = 0$.

The exact sequence of a closed immersion $X_t \subset \tilde{U}_t$ (proposition 1.14) is of the form

$$K^{\mathbb{T}}(X_t) \xrightarrow{i_*} K^{\mathbb{T}}(\tilde{U}_t) \xrightarrow{j^*} K^{\mathbb{T}}(\tilde{U}_{t-1}) \longrightarrow 0\,.$$

Consider the element $a_{|\tilde{U}_t} \in K^{\mathbb{T}}(\tilde{U}_t)$. By the inductive assumption $j^* a_{|\tilde{U}_t} = 0$, thus $a_{|\tilde{U}_t} = i_*\alpha$. Equations (14) imply that

$$0 = a_{|X_t} = i^* a_{|\tilde{U}_t} = i^* i_* \alpha = \alpha \cdot eu(X_t \subset \tilde{U}_t) \in K^{\mathbb{T}}(X_t)\,.$$

If $t < r$ then the ring $K^{\mathbb{T}}(X_t)$ is a domain. If $t \geq r$ then

$$eu(X_t \subset \tilde{U}_t) = eu(U_{\mathbf{e}_t} \subset \tilde{U}_t) = \pi^*_{\mathbf{e}_t} eu(M^+_{\mathbf{e}_t} \subset M)$$

is not a zero divisor (lemma 5.6). It follows that in both cases $\alpha = 0$ and

$$a_{|\tilde{U}_t} = i_*\alpha = 0.$$

□

5.2. Sufficient condition

In this section we present a condition sufficient to prove that the twisted motivic Chern class of BB-cell satisfies the support axiom. Let us remind our assumptions. M is a projective, smooth A-variety and $X = T^*M$ is the cotangent variety with the induced $\mathbb{T}$-action (see example 1.61). We denote by $\pi\colon X \to M$ the $\mathbb{T}$-equivariant projection. Suppose that the fixed point set M^A is finite. Let $\sigma\colon \mathbb{C}^* \to A$ be a good one parameter subgroup (see definition 1.62).

Definition 5.7. We say that the pair (M, σ) satisfies the local product property if:

- For any fixed point $\mathbf{e} \in M^A$ there exist an A-equivariant Zariski open neighbourhood $U_{\mathbf{e}}$ of $\mathbf{e}$, a smooth A-variety $Z_{\mathbf{e}}$ (called slice) and an A-equivariant isomorphism
$$\theta_{\mathbf{e}}\colon U_{\mathbf{e}} \simeq (U_{\mathbf{e}} \cap M^+_{\mathbf{e}}) \times Z_{\mathbf{e}}\,.$$
- For any fixed point $\mathbf{e}' \in M^A$ there exist an invariant subvariety $Z'_{\mathbf{e},\mathbf{e}'} \subset Z_{\mathbf{e}}$ such that $\theta_{\mathbf{e}}$ induces an isomorphism:
$$U_{\mathbf{e}} \cap M^+_{\mathbf{e}'} \simeq (U_{\mathbf{e}} \cap M^+_{\mathbf{e}}) \times Z'_{\mathbf{e},\mathbf{e}'}\,.$$

The local product property is a very strong condition. Nevertheless, there are important examples of spaces having this property, such as homogeneous varieties (cf. Theorem 5.15).

PROPOSITION 5.8. *Suppose that the pair (M,σ) satisfies the local product property. Then the pair (M,σ) is admisible (definition 1.64).*

LEMMA 5.9. *Let B, C be smooth varieties. Consider a point $b\in B$ and a locally closed smooth subvariety $\tilde{B}\subset B$. Then*

$$T^*(C\times B)_{|C\times b}\cap \overline{\nu^*(C\times \tilde{B}\subset C\times B)}\subset \nu^*(C\times b\subset C\times B)\,.$$

PROOF. It follows from identifications

$$\begin{aligned}&\overline{\nu^*(C\times \tilde{B}\subset C\times B)}=C\times \overline{\nu^*(\tilde{B}\subset B)}\,,\\ &T^*(C\times B)_{|C\times b}=T^*C\times T^*_bB\,,\\ &\nu^*(C\times b\subset C\times B)=C\times T^*_bB\,.\end{aligned}$$

□

PROOF OF PROPOSITION 5.8. We need to show that the subvariety $\bigsqcup X^+_{\mathbf{e}}$ is closed in X. It is enough to prove that for a pair of fixed points $\mathbf{e}>\mathbf{e}'$ we have

$$\overline{X^+_{\mathbf{e}}}\cap T^*M_{|M^+_{\mathbf{e}'}}\subset X^+_{\mathbf{e}'}\,. \tag{15}$$

These subsets are A-equivariant and the variety $M^+_{\mathbf{e}'}$ is contracted to a point by the one parameter subgroup σ. Thus, it is enough to show that inclusion (15) holds after intersection with $T^*M_{|U}$ for some open neighbourhood $\mathbf{e}'\in U\subset M$. Take $U_{\mathbf{e}'}$ from the definition of local product property as a neighbourhood of $\mathbf{e}'$. Then, the desired inclusion follows from lemma 5.9 for

$$B=Z_{\mathbf{e}'}\,,\qquad C=M^+_{\mathbf{e}'}\cap U_{\mathbf{e}'}\,,\qquad b=Z'_{\mathbf{e}',\mathbf{e}'}\,,\qquad \tilde{B}=Z'_{\mathbf{e}',\mathbf{e}}\,.$$

□

PROPOSITION 5.10. *Suppose that the pair (M,σ) satisfies the local product property. Let $\mathbf{e}\in M^A$ be a fixed point. Then*

- *The motivic Chern class of BB-cell satisfies the support axiom for stab$(\mathbf{e})$, i.e.*
$$\operatorname{supp}(\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(\mathrm{mC}^A_y(M^+_{\mathbf{e}}\to M)))\subset \bigcup_{\mathbf{e}'\le\mathbf{e}} X^+_{\mathbf{e}'}$$
- *Let $s\in Pic(M)\otimes_{\mathbb{Z}}\mathbb{Q}$ be a slope and $\Delta_{\mathbf{e},s}$ the divisor defined in section 4.2. Denote by $i_{\mathbf{e}}\colon \overline{M^+_{\mathbf{e}}}\to M$ the inclusion of the closure of the BB-cell. The twisted motivic Chern class of BB-cell satisfies the support axiom, i.e.*
$$\operatorname{supp}\left(\mathfrak{h}^{-\dim(M^+_{\mathbf{e}})}\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}^A_y(\overline{M^+_{\mathbf{e}}},\partial M^+_{\mathbf{e}};\Delta_{\mathbf{e},s}))\right)\subset \bigcup_{\mathbf{e}'\le\mathbf{e}} X^+_{\mathbf{e}'}$$

LEMMA 5.11. *Suppose that the pair (M,σ) satisfies the local product property. Let $\mathbf{e},\mathbf{e}'\in M^A$ be a pair of fixed points. Let $U_{\mathbf{e}'}$ be an open neighbourhood of $\mathbf{e}'$ as in the local product property definition. Let D be an invariant $\mathbb{Q}$-Cartier divisor on $\overline{M^+_{\mathbf{e}}}$ with support contained in the boundary $\partial\overline{M^+_{\mathbf{e}}}$. Denote by*

$$Y:=U_{\mathbf{e}'}\cap M^+_{\mathbf{e}'}\,,\qquad \partial\overline{Z'_{\mathbf{e}',\mathbf{e}}}:=\overline{Z'_{\mathbf{e}',\mathbf{e}}}\setminus Z'_{\mathbf{e}',\mathbf{e}}\,.$$

Then, there exists a $\mathbb{Q}$-Cartier divisor D' on the variety $\overline{Z'_{\mathbf{e}',\mathbf{e}}}$ with support in the boundary, such that $\theta_{\mathbf{e}'}$ induces an equality

$$\mathrm{mC}^A_y(\overline{M^+_{\mathbf{e}}}\cap U_{\mathbf{e}'},\partial\overline{M^+_{\mathbf{e}}}\cap U_{\mathbf{e}'};D\cap U_{\mathbf{e}'})=\theta^*_{\mathbf{e}'}\,\mathrm{mC}^A_y(Y\times\overline{Z'_{\mathbf{e}',\mathbf{e}}},Y\times\partial\overline{Z'_{\mathbf{e}',\mathbf{e}}};Y\times D')\,.$$

PROOF. We need to prove that

$$\theta_{\mathbf{e}'}(\overline{M_{\mathbf{e}}^{+}} \cap U_{\mathbf{e}'}) = Y \times \overline{Z'_{\mathbf{e}',\mathbf{e}}}\,, \tag{16}$$

$$\theta_{\mathbf{e}'}(\partial\overline{M_{\mathbf{e}}^{+}} \cap U_{\mathbf{e}'}) = Y \times \partial\overline{Z'_{\mathbf{e}',\mathbf{e}}}\,, \tag{17}$$

$$\theta_{\mathbf{e}'}(D \cap U_{\mathbf{e}'}) = Y \times D'\,. \tag{18}$$

For the first equality (16) note that

$$\theta_{\mathbf{e}'}\left(U_{\mathbf{e}'} \cap \overline{M_{\mathbf{e}}^{+}}\right) = \theta_{\mathbf{e}'}\left(\overline{U_{\mathbf{e}'} \cap M_{\mathbf{e}}^{+}}\right) = \overline{\theta_{\mathbf{e}'}(U_{\mathbf{e}'} \cap M_{\mathbf{e}}^{+})} = \overline{Y \times Z'_{\mathbf{e}',\mathbf{e}}} = Y \times \overline{Z'_{\mathbf{e}',\mathbf{e}}}\,,$$

where the first closure is taken in M, the second in $U_{\mathbf{e}'}$, the third and the fourth in $Y \times Z_{\mathbf{e}'}$ and the fifth in the slice $Z_{\mathbf{e}'}$. By definition the isomorphism $\theta_{\mathbf{e}'}$ satisfies

$$\theta_{\mathbf{e}'}(U_{\mathbf{e}'} \cap M_{\mathbf{e}}^{+}) = Y \times Z'_{\mathbf{e}',\mathbf{e}}\,.$$

Taking the set difference with equation (16) we obtain (17).
Equality (18) follows from (17). Namely, the variety Y is irreducible, thus the irreducible components of

$$\theta_{\mathbf{e}'}(\partial\overline{M_{\mathbf{e}}^{+}} \cap U_{\mathbf{e}'}) = Y \times \partial\overline{Z'_{\mathbf{e}',\mathbf{e}}}$$

are of the product form. The divisor D is supported on $\partial\overline{M_{\mathbf{e}}^{+}} \cap U_{\mathbf{e}'}$, therefore it is a formal sum of irreducible components of $\partial\overline{M_{\mathbf{e}}^{+}} \cap U_{\mathbf{e}'}$. □

PROOF OF PROPOSITION 5.10. The twisted motivic Chern class for a small antiample slope coincides with the standard motivic Chern class (see remark 3.7). Therefore, it is enough to prove the proposition for the twisted motivic Chern class. The equivalent statement of the support axiom (proposition 5.2) implies that it is enough to prove that

$$\left(\pi^*\rho(i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^{+}}, \partial\overline{M_{\mathbf{e}}^{+}}; \Delta_{\mathbf{e},s}))\right)_{|\mathbf{e}'}$$

is divisible by $\lambda_{-\mathfrak{h}}((T^{+}_{\mathbf{e}'}M)^*)$ when $\mathbf{e}' \leq \mathbf{e}$ and equal to zero in the other case.
By the exact sequence of a closed immersion $\overline{M_{\mathbf{e}}^{+}} \subset M$ the considered class can be nonzero only for $\mathbf{e}' \leq \mathbf{e}$. Let us focus on this case. The map ρ is defined by $\rho(y) = -\mathfrak{h}$, thus it is enough to prove that

$$\lambda_y((T^{+}_{\mathbf{e}'}M)^*) \mid i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^{+}}, \partial\overline{M_{\mathbf{e}}^{+}}; \Delta_{\mathbf{e},s})_{|\mathbf{e}'}\,.$$

Let $U_{\mathbf{e}'}$ be an open neighbourhood of $\mathbf{e}'$ as in the local product property definition. Denote by

$$\tilde{i}_{\mathbf{e}}\colon \overline{M_{\mathbf{e}}^{+}} \cap U_{\mathbf{e}'} \to U_{\mathbf{e}'}\,, \qquad Y := U_{\mathbf{e}'} \cap M_{\mathbf{e}'}^{+}\,, \qquad \iota\colon \overline{Z'_{\mathbf{e}',\mathbf{e}}} \hookrightarrow Z_{\mathbf{e}'}\,.$$

Then

$$\begin{aligned}
i_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^{+}}, \partial\overline{M_{\mathbf{e}}^{+}}; \Delta_{\mathbf{e},s})_{|\mathbf{e}'} &= \tilde{i}_{\mathbf{e}*}\,\mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^{+}} \cap U_{\mathbf{e}'}, \partial\overline{M_{\mathbf{e}}^{+}} \cap U_{\mathbf{e}'}; \Delta_{\mathbf{e},s} \cap U_{\mathbf{e}'})_{|\mathbf{e}'} \\
&= \tilde{i}_{\mathbf{e}*}\,\mathrm{mC}_y^A(Y \times \overline{Z'_{\mathbf{e}',\mathbf{e}}}, Y \times \partial Z'_{\mathbf{e}',\mathbf{e}}; Y \times D)_{|\mathbf{e}'} \\
&= \tilde{i}_{\mathbf{e}*}\left(\mathrm{mC}_y^A(id_Y) \boxtimes \mathrm{mC}_y^A(\overline{Z'_{\mathbf{e}',\mathbf{e}}}, \partial Z'_{\mathbf{e}',\mathbf{e}}; D)\right)_{|\mathbf{e}'} \\
&= \mathrm{mC}_y^A(id_Y)_{|\mathbf{e}'} \cdot \iota_*\left(\mathrm{mC}_y^A(\overline{Z'_{\mathbf{e}',\mathbf{e}}}, \partial Z'_{\mathbf{e}',\mathbf{e}}; D)\right)_{|\mathbf{e}'} \\
&= \lambda_y((T^{+}_{\mathbf{e}'}M)^*) \cdot \iota_*\left(\mathrm{mC}_y^A(\overline{Z'_{\mathbf{e}',\mathbf{e}}}, \partial Z'_{\mathbf{e}',\mathbf{e}}; D)\right)_{|\mathbf{e}'}\,.
\end{aligned}$$

The first equality follows from proposition 3.9 (1), the second from lemma 5.11 and the third from proposition 3.9 (2). The fourth is a consequence of $\tilde{i}_{\mathbf{e}} = id_Y \times \iota$ and the fifth follows from the fact that Y is a smooth variety. □

THEOREM 5.12. *Let M be a smooth, projective A-variety and $\sigma \colon \mathbb{C}^* \to A$ a good one parameter subgroup. Suppose that the fixed point set M^A is finite and the pair (M, σ) satisfies the local product property. Consider the cotangent variety $X = T^*M$ with the induced $A \times \mathbb{C}^*$-action. Denote by $\pi \colon X \to M$ the projection. For a fixed point $\mathbf{e} \in M^A$ let $i_{\mathbf{e}} \colon \overline{M_{\mathbf{e}}^+} \subset M$ be an inclusion. Let $s \in Pic(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a slope. Then, the stable envelopes for the variety X are equal to the twisted motivic Chern classes of BB-cells of M i.e.*

$$\mathfrak{h}^{-\dim(M_{\mathbf{e}}^+)} \pi^* \rho \left(i_{\mathbf{e}*} \, \mathrm{mC}_y^A(\overline{M_{\mathbf{e}}^+}, \partial \overline{M_{\mathbf{e}}^+}; \Delta_{\mathbf{e},s}) \right) = Stab^s(\mathbf{e}) \,.$$

PROOF. By theorem 4.2 (or theorem 2.2 for a trivial slope) the considered twisted motivic Chern class satisfies all but one axioms of the stable envelope. By proposition 5.10 the local product property of (M, σ) implies that it satisfies also the support axiom. □

COROLLARY 5.13. *Consider the situation as in theorem 5.12. Suppose that the slope s is trivial or small antiample. Then*

$$\mathfrak{h}^{-\dim(M_{\mathbf{e}}^+)} \pi^* \rho \left(\mathrm{mC}_y^A(M_{\mathbf{e}}^+ \to M) \right) = Stab^s(\mathbf{e}) \,.$$

5.3. Homogenous varieties

Let G be a reductive, complex Lie group with a chosen maximal torus A and a Borel subgroup B^+. Any one parameter subgroup $\sigma \colon \mathbb{C}^* \to A$ induces a linear functional

$$\varphi_\sigma \colon \mathfrak{a}^* \to \mathbb{C}.$$

For a general enough subgroup σ we can assume that no roots belong to the kernel of this functional. Consider the Borel subgroups B_σ^+ such that the corresponding Lie algebra is the union of these weight spaces whose characters are non-negative with respect to φ_σ. Denote its unipotent subgroup by U_σ^+. Analogously, one can define groups B_σ^- and U_σ^-.

For a parabolic group $B^+ \subset P \subset G$ we consider the homogenous variety G/P with the induced action of the torus A. The fixed point set $(G/P)^A$ is finite. It is a classical fact that the positive (respectively negative) BB-cells with respect to σ are orbits of the group B_σ^+ (respectively B_σ^-). These orbits are called Schubert cells. In this section we prove that the pair $(G/P, \sigma)$ satisfies the local product property.

REMARK 5.14. For an arbitrary one parameter subgroup σ the maximal torus A is contained in B_σ i.e. $A \subset B_\sigma^+$. In general the group B_σ^+ may be not contained in P.

THEOREM 5.15. *The pair $(G/P, \sigma)$ satisfies the local product property for general enough σ. For any fixed point $\mathbf{e} \in (G/P)^A$ there is an A-invariant open neighbourhood $U_{\mathbf{e}}$ such that:*

(1) There exists an A-equivariant isomorphism

$$\theta_{\mathbf{e}} \colon U_{\mathbf{e}} \simeq \left(U \cap (G/P)_{\mathbf{e}}^+ \right) \times \left(U_{\mathbf{e}} \cap (G/P)_{\mathbf{e}}^- \right)$$

(2) *For any fixed point* $\mathbf{e}' \in (G/P)^A$ *the isomorphism* $\theta_{\mathbf{e}}$ *induces an isomorphism:*

$$U_{\mathbf{e}} \cap (G/P)^+_{\mathbf{e}'} \simeq \left(U \cap (G/P)^+_{\mathbf{e}}\right) \times \left(U \cap (G/P)^-_{\mathbf{e}} \cap (G/P)^+_{\mathbf{e}'}\right)$$

COROLLARY 5.16. *Consider the homogenous variety* G/P *described above. The stable envelopes for the cotangent bundle* T^*G/P *are equal to the twisted motivic Chern classes of the Schubert cells, explicitly*

$$\mathfrak{h}^{-\dim((G/P)^+_{\mathbf{e}})}\pi^*\rho\left(i_{\mathbf{e}*}\,\mathrm{mC}^A_y(\overline{(G/P)^+_{\mathbf{e}}}, \partial\overline{(G/P)^+_{\mathbf{e}}}; \Delta_{\mathbf{e},s})\right) = Stab^s(\mathbf{e})\,.$$

For the trivial slope $s = \theta$ *or a small antiample slope we get*

$$\mathfrak{h}^{-\dim((G/P)^+_{\mathbf{e}})}\pi^*\rho\left(mC^A_y((G/P)^+_{\mathbf{e}} \to G/P)\right) = Stab^s(\mathbf{e})\,.$$

PROOF. It is an immediate consequence of theorems 5.12 and 5.15. □

REMARK 5.17. For $M = G/B$, our result for a small anti-ample slope agrees with the previous results of [**AMSS19**, Theorem 8.5 and Remark 8.7] up to a change of $\mathfrak{h}$ to $\mathfrak{h}^{-1}$. This difference is a consequence of the fact that in [**AMSS19**] the inverse action of the factor $\mathbb{C}^*$ is considered.

In the course of proof we use the following interpretation of classical notions of the theory of Lie groups in the language of BB-decomposition.

LEMMA 5.18. *Consider the action of the one parameter subgroup* σ *on the group* G *defined by conjugation. Denote by* F *the component of the fixed point set* G^σ *which contains the identity. For a subset* $Y \subset F$. *Let*

$$G^+(Y) = \{x \in G |\lim_{t\to 0}\sigma(t)\cdot x \in Y\} \text{ and } G^-(Y) = \{x \in G |\lim_{t\to \infty}\sigma(t)\cdot x \in Y\}$$

be the preimages of Y *in the* A*-invariant projections* $G^+_F \to F$ *and* $G^-_F \to F$, *respectively.*

(1) *The Borel subgroup* B^+_σ *(respectively* B^-_σ*) is the preimage of the maximal torus* A *in the projection* $G^+_F \to F$ *(respectively* $G^-_F \to F$*) i.e.*

$$B^+_\sigma = G^+(A)\,.$$

(2) *The unipotent subgroup* U^+_σ *(respectively* U^-_σ*) is the fiber of the projection* $G^+_F \to F$ *(respectively* $G^-_F \to F$*) over the identity element i.e.*

$$U^+_\sigma = G^+(id)\,.$$

PROOF. We prove only the first case for the positive Borel subgroup. The other cases are analogous. It is enough to show that $G^+(A)$ is a connected subgroup of G whose Lie algebra coincides with the Lie algebra of B^+_σ.

The subvariety $G^+(A)$ is a subgroup of G because the maximal torus A is a group and the limit preserves multiplication, i.e. for $g, h \in G^+(A)$ we have

$$\begin{aligned}\lim_{t\to 0}\sigma(t)\cdot gh^{-1} &= \lim_{t\to 0}\sigma(t)gh^{-1}\sigma(t)^{-1} = \lim_{t\to 0}\sigma(t)g\sigma(t)^{-1}\sigma(t)h^{-1}\sigma(t)^{-1} \\ &= \lim_{t\to 0}(\sigma(t)\cdot g)(\sigma(t)\cdot h)^{-1} \\ &= \left(\lim_{t\to 0}\sigma(t)\cdot g\right)\left(\lim_{t\to 0}\sigma(t)\cdot h\right)^{-1} \in A\,.\end{aligned}$$

Moreover, the variety $G^+(A)$ is connected because the maximal torus A is connected. So it is enough to compute the tangent space to $G^+(A)$ at identity. By theorem 1.56 (1) it is equal to the nonnegative part of the σ-representation $\mathfrak{g}$. This is exactly the Lie algebra of the Borel subgroup B^+_σ. □

PROOF OF THE THEOREM 5.15. Note that the Weyl group acts transitively on the fixed point set $(G/P)^A$. Thus, replacing the subgroup σ by its conjugate by a Weyl group element, we may assume that a fixed point $\mathbf{e}$ is equal to the class of identity.

Let $\mathfrak{p} \subset \mathfrak{g}$ be the Lie subalgebra of the parabolic subgroup P. Denote by $\mathfrak{u}_P$ the Lie subalgebra consisting of the root spaces which do not belong to $\mathfrak{p}$. Let U_P be the corresponding Lie group. The group U_P is unipotent (as a subgroup of the unipotent group U^-). Consider the action of the torus A on U_P given by conjugation. Let us note two facts from the theory of Lie groups.

(1) U_P is isomorphic to its complex Lie algebra as a complex A-variety (cf. [**Bor91**, Paragraph 15.3b], or [**KMT74**, Paragraph 8.0]).
(2) The quotient map $p\colon G \to G/P$ induces an A-equivariant isomorphism from U_P to some open neighbourhood of identity.

Choose $p(U_P)$ as a neighbourhood of the class of identity. Let

$$X_+ := p(U_P) \cap (G/P)^+_{id}; \qquad X_- := p(U_P) \cap (G/P)^-_{id}$$

The second observation and the second point of lemma 5.18 imply that:

$$X_+ := p(U_P) \cap (G/P)^+_{id} \simeq U_P \cap G^+(id) \simeq U_P \cap U^+_\sigma \,,$$

analogously

$$X_- := p(U_P) \cap (G/P)^-_{id} \simeq U_P \cap G^-(id) \simeq U_P \cap U^-_\sigma \,.$$

Both isomorphisms are given by the quotient morphism $p\colon G \to G/P$. We define a morphism

$$\theta_{id}\colon X_+ \times X_- \to p(U_P)$$

as the multiplication in U_P. We aim to prove that this is an isomorphism. We start by showing injectivity on points. Both varieties X_+ and X_- are subgroups of U_P. So to prove injectivity it is enough to show that $X_+ \cap X_- = \{id\}$. But X_+ is contained in the positive unipotent group and X_- in the negative unipotent group, so their intersection must be trivial.

As a variety U_P is isomorphic to an affine space - its Lie algebra $\mathfrak{u}_P$. The induced action of A on the linear space $\mathfrak{u}_P$ is linear as a part of the adjoint representation of G. It follows that both X_+ and X_- are BB-cells of a linear action on a linear space and therefore linear subspaces. Thus, the product $X_+ \times X_-$ is isomorphic to an affine space of dimension equal to dimension of U_P. Therefore, the map θ_{id} is an algebraic endomorphism of an affine space which is injective on points. The Ax–Grothendieck theorem ([**Ax68**] or [**Gro66**, Theorem 10.4.11.]) implies that it is bijective on points. All affine spaces are smooth and connected so the Zariski main theorem [**Gro61**, Theorem 4.4.3] implies that θ_{id} is an algebraic isomorphism.

To prove the second property it is enough to show the containment

$$\theta_{id}\left(X_+ \times (p(U_P) \cap (G/P)^+_{\mathbf{e}'})\right) \subset (G/P)^+_{\mathbf{e}'},$$

for any fixed point $\mathbf{e}' \in (G/P)^A$. Note that

$$X_+ \subset U_\sigma^+ \subset B_\sigma^+.$$

Moreover, the BB-cell $(G/P)_y^+$ is an orbit of the group B_σ^+ and the morphism θ coincides with the action of B_σ^+. So the desired inclusion holds. $\square$

CHAPTER 6

Example: Lagrangian Grassmanian $LG(2,4)$

6.1. Description of $\mathrm{LG}(2,4)$

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be the standard basis of a vector space $\mathbb{C}^4$. Let ω be a symplectic form given by matrix

$$\omega = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

We consider Grassmanian of Lagrangian subspaces in $\mathbb{C}^4$ i.e.

$$\mathrm{LG}(2,4) := \{V \in Gr(2,4) \mid \omega_{|V} = 0\}.$$

This is a homogenous variety of the symplectic group $\mathrm{Sp}(2)$. The maximal torus $A \subset \mathrm{Sp}(2)$ consists of diagonal matrices which preserve the form ω i.e. of matrices of the form

$$\begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s^{-1} \end{bmatrix}$$

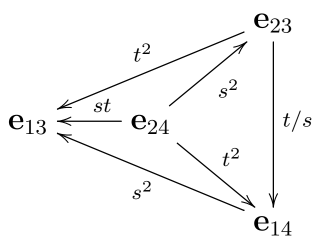
6.1.1. Fixed points and tangent weights. The fixed point set LG^A is finite. It consists of coordinate Lagrangian subspaces

$$\mathrm{LG}^A = \{\mathrm{span}(\mathbf{v}_2, \mathbf{v}_4); \mathrm{span}(\mathbf{v}_2, \mathbf{v}_3); \mathrm{span}(\mathbf{v}_1, \mathbf{v}_4); \mathrm{span}(\mathbf{v}_1, \mathbf{v}_3)\}.$$

Let $\mathbf{e}_{ij} := \mathrm{span}(\mathbf{v}_i, \mathbf{v}_j)$. Tangent weights at the fixed point set are collected in the following table

fixed point	weights
$\mathbf{e}_{24}$	$t^2,\ s^2,\ ts$
$\mathbf{e}_{23}$	$t^2,\ 1/s^2,\ t/s$
$\mathbf{e}_{14}$	$1/t^2,\ s^2,\ s/t$
$\mathbf{e}_{13}$	$1/t^2,\ 1/s^2,\ 1/ts$

The GKM graph is of the form (arrows point from the positive weight to the negative one)



6.1.2. Schubert varieties. Let $\sigma : \mathbb{C}^* \to A$ be a one parameter subgroup given by

$$\sigma(t) = \operatorname{diag}\{t^2, t^{-2}, t, t^{-1}\}\,.$$

We consider the corresponding BB-decomposition. The BB-order is given by

$$\mathbf{e}_{24} > \mathbf{e}_{23} > \mathbf{e}_{14} > \mathbf{e}_{13}\,.$$

We denote the closure of BB-cell of $\mathbf{e}_{ij}$ by LG_{ij}, i.e.

$$\mathrm{LG}_{ij} := \overline{\mathrm{LG}^+_{\mathbf{e}_{ij}}}\,.$$

The variety LG_{13} is a single point and LG_{24} is the whole Lagrangian Grassmanian. The variety LG_{14} is isomorphic to $\mathbb{P}^1$. The pairs $(\mathrm{LG}_{ij}, \partial\,\mathrm{LG}_{ij})$ are of the form

$$\begin{aligned}
(\mathrm{LG}_{24}, \partial\,\mathrm{LG}_{24}) &= (\mathrm{LG}(2,4), \mathrm{LG}_{23})\,,\\
(\mathrm{LG}_{23}, \partial\,\mathrm{LG}_{23}) &= (\mathrm{LG}_{23}, \mathrm{LG}_{14})\,,\\
(\mathrm{LG}_{14}, \partial\,\mathrm{LG}_{14}) &= (\mathrm{LG}_{14}, \mathrm{LG}_{13}) \simeq (\mathbb{P}^1, \{0\})\,,\\
(\mathrm{LG}_{13}, \partial\,\mathrm{LG}_{13}) &= (\mathrm{LG}_{13}, \varnothing) = (\mathbf{e}_{13}, \varnothing)\,.
\end{aligned}$$

6.1.3. SNC resolutions. The varieties $\mathrm{LG}_{13}, \mathrm{LG}_{14}$ and LG_{24} are smooth. The variety LG_{23} has one singular point $\mathbf{e}_{13}$. The point $\mathbf{e}_{13}$ has an open neighbourhood U isomorphic to an affine space $\mathbb{C}^3$. A point $(a,b,c) \in \mathbb{C}^3$ corresponds to the Lagrangian subspace spanned by rows of the matrix

$$\begin{bmatrix} 1 & a & 0 & c \\ 0 & -c & 1 & b \end{bmatrix}$$

In these coordinates the variety $\mathrm{LG}_{23} \cap U$ is given by the equation

$$c^2 + ab = 0\,.$$

The pairs $(\mathrm{LG}_{13}, \partial\,\mathrm{LG}_{13})$ and $(\mathrm{LG}_{14}, \partial\,\mathrm{LG}_{14})$ are already SNC. We need to find resolution of two remaining pairs. Let Z be the blow up of $\mathrm{LG}(2,4)$ at the point $\mathbf{e}_{13}$. Denote by E its exceptional divisor. The preimage of the neighbourhood U is the blow up of an affine space in a point. Thus, it may be covered by three A-invariant open subsets U_a, U_b, U_c isomorphic to $\mathbb{C}^3$. On these neighbourhoods we have standard choices of coordinates $(a, \tilde{b}_a, \tilde{c}_a)$, $(\tilde{a}_b, b, \tilde{c}_b)$, $(\tilde{a}_c, \tilde{b}_c, c)$ such that

$$x \cdot \tilde{y}_x = y\,,$$

for $x, y \in \{a,b,c\}$.

Consider the projection $\pi\colon Z \to \mathrm{LG}(2,4)$. Let Z_{23} be the proper transform of LG_{23}. The total transform is equal to

$$\pi^{-1}(\mathrm{LG}_{23}) = Z_{23} \cup E\,.$$

Computations in local coordinates show that these two subvarieties form a SNC divisor. Therefore, $(Z, \pi^{-1}(\mathrm{LG}_{23}))$ is a SNC resolution of $(\mathrm{LG}(2,4), \mathrm{LG}_{23})$. Denote the interior of Z by

$$Z^o = Z \setminus (Z_{23} \cup E)\,.$$

Consider the projection $p\colon Z_{23} \to \mathrm{LG}(2,4)$. It is the blow up of LG_{23} in the singular point $\mathbf{e}_{13}$. Let Z_{14} be the proper transform of LG_{14} and E' the exceptional divisor of p. The total transform is equal to

$$p^{-1}(\mathrm{LG}_{13}) = Z_{14} \cup E'\,.$$

Computations in local coordinates show that these two subvarieties form a SNC divisor. Therefore, $(Z_{23}, p^{-1}(\mathrm{LG}_{14}))$ is a SNC resolution of $(\mathrm{LG}_{23}, \mathrm{LG}_{14})$. Denote the interior of Z_{23} by

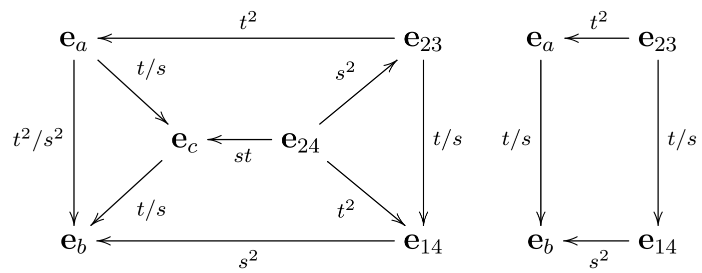
$$Z_{23}^o = Z \setminus (Z_{14} \cup E')\,.$$

Remark 6.1. The variety $\mathrm{LG}_{23} \cap U$ is an affine cone over quadric $ab + c^2 = 0$. The variety Z_{23} is the blow up at the vertex of this cone.

All tangent weights at $\mathbf{e}_{13}$ have multiplicity one therefore fixed point sets Z^A and Z_{23}^A are finite. Let $\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c$ denote points corresponding to $(0,0,0)$ in U_a, U_b and U_c respectively. We have

$$Z^A = \{\mathbf{e}_{24}, \mathbf{e}_{23}, \mathbf{e}_{14}, \mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c\}\,,$$
$$Z_{23}^A = \{\mathbf{e}_{23}, \mathbf{e}_{14}, \mathbf{e}_a, \mathbf{e}_b\}\,.$$

The GKM graphs of Z and Z_{23} are of the form



Remark 6.2. The variety Z is not a GKM space (see [**GKM98**] or [**AF21**, discussion after Corrolary 4.3]), there is a family of one dimensional A–orbits between $\mathbf{e}_a$ and $\mathbf{e}_b$. Normal weight to the subvariety Z_{23} at $\mathbf{e}_a$ is t^2/s^2.

6.1.4. Picard Group. Lagrangian Grassmanian is a Fano variety, therefore

$$Pic(\mathrm{LG}(2,4)) \simeq H^2(\mathrm{LG}(2,4), \mathbb{Z})\,.$$

The Białynicki-Birula theorem [**BB73**, Theorem 4.3] implies that the group $H^2(\mathrm{LG}(2,4), \mathbb{Z})$ is the free abelian group generated by the class of LG_{23}. Therefore

$$Pic(\mathrm{LG}(2,4)) \simeq \mathbb{Z}\,,$$

is generated by the class of line bundle $\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})$. Consider the natural A-linearisation of $\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})$. Let us compute its weights at the fixed point set. The point $\mathbf{e}_{24}$ does not belong to LG_{23}, therefore the weight at $\mathbf{e}_{24}$ is trivial. Weights at smooth points $\mathbf{e}_{23}$ and $\mathbf{e}_{14}$ are normal weights to the subvariety LG_{23}. The remaining weight at $\mathbf{e}_{13}$ can be computed using restrictions to one dimensional orbits. It is of the form $t^a s^b$. Restricting to the orbit connecting $\mathbf{e}_{24}$ and $\mathbf{e}_{13}$ we acquire $a = b$ and restricting to the orbit connecting $\mathbf{e}_{23}$ and $\mathbf{e}_{13}$ we obtain $a=-2$. To conclude the weights are given by

	$\mathbf{e}_{24}$	$\mathbf{e}_{23}$	$\mathbf{e}_{14}$	$\mathbf{e}_{13}$
$\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})$	1	$1/s^2$	$1/t^2$	$1/t^2s^2$

Analogously we compute the weights of line bundles $\mathcal{O}_{\mathrm{LG}_{14}}(\mathrm{LG}_{13})$, $\mathcal{O}_Z(E)$, $\mathcal{O}_Z(Z_{23})$, $\mathcal{O}_{Z_{23}}(Z_{E'})$ and $\mathcal{O}_{Z_{23}}(Z_{14})$.

	$\mathbf{e}_{14}$	$\mathbf{e}_{13}$
$\mathcal{O}_{\mathrm{LG}_{14}}(\mathrm{LG}_{13})$	1	$1/s^2$

	$\mathbf{e}_{23}$	$\mathbf{e}_{14}$	$\mathbf{e}_a$	$\mathbf{e}_b$
$\mathcal{O}_{Z_{23}}(Z_{14})$	1	s/t	1	s/t
$\mathcal{O}_{Z_{23}}(E')$	1	1	$1/t^2$	$1/s^2$

	$\mathbf{e}_{14}$	$\mathbf{e}_{23}$	$\mathbf{e}_{14}$	$\mathbf{e}_a$	$\mathbf{e}_b$	$\mathbf{e}_c$
$\mathcal{O}_Z(Z_{23})$	1	$1/s^2$	$1/t^2$	t^2/s^2	s^2/t^2	1
$\mathcal{O}_Z(E)$	1	1	1	$1/t^2$	$1/s^2$	$1/ts$

The localization theorem (Theorem 1.22) implies that

$$\begin{aligned}
\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})_{|\,\mathrm{LG}_{13}} &\simeq \mathbb{C}_{1/s^2t^2}\,,\\
\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})_{|\,\mathrm{LG}_{14}} &\simeq \mathcal{O}_{\mathrm{LG}_{14}}(\mathrm{LG}_{13})\otimes\mathbb{C}_{1/t^2}\,,\\
p^*(\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})_{|\,\mathrm{LG}_{23}}) &\simeq \mathcal{O}_{Z_{23}}(2Z_{14}+E')\otimes\mathbb{C}_{1/s^2}\,,\\
\pi^*\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23}) &\simeq \mathcal{O}_Z(Z_{23}+2E)\,.
\end{aligned}$$

Therefore, in non-equivariant Picard groups we have equalities

$$\begin{aligned}
\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})_{|\,\mathrm{LG}_{13}} &\simeq \mathcal{O}_{LG_{13}}\,,\\
\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})_{|\,\mathrm{LG}_{14}} &\simeq \mathcal{O}_{\mathrm{LG}_{14}}(\mathrm{LG}_{13})\,,\\
p^*(\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})_{|\,\mathrm{LG}_{23}}) &\simeq \mathcal{O}_{Z_{23}}(2Z_{14}+E')\,,\\
\pi^*\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23}) &\simeq \mathcal{O}_Z(Z_{23}+2E)\,.
\end{aligned}$$

6.2. Twisted classes

In this section we collect computed twisted motivic Chern classes of Schubert cells in $\mathrm{LG}(2,4)$. We present detailed computation only for the class of a Schubert variety LG_{23}. In our opinion this is the most complicated class.
Denote by ι_{ij} the inclusion of Schubert variety

$$\iota_{ij}\colon\ \mathrm{LG}_{ij}\subset\mathrm{LG}(2,4)\,.$$

Let $\lambda\in\mathbb{Q}$ be a rational number. We want to compute classes

$$\iota_{ij*}\,\mathrm{mC}^A_y(\mathrm{LG}_{ij},\partial\,\mathrm{LG}_{ij};\Delta_\lambda)\in K^A(\mathrm{LG}(2,4))[y]$$

where Δ_λ is a $\mathbb{Q}$-Cartier divisor corresponding to $\mathcal{O}_{\mathrm{LG}(2,4)}(\lambda\cdot\mathrm{LG}_{23})_{|LG_{ij}}$ (see section 4.2). For simplicity we denote the above class by $\mathrm{mC}^A_y(ij,\lambda)$.

6.2.1. Results. Let $\alpha=2\lceil\lambda\rceil-\lceil2\lambda\rceil$. Note that $\alpha=1$ when $\lceil\lambda\rceil-\lambda\geq1/2$ and $\alpha=0$ in the other case.

For the point $\mathbf{e}_{13}$ we have

$$\begin{aligned}
\mathrm{mC}_y^A(13,\lambda)_{|\mathbf{e}_{24}} &= 0\,,\\
\mathrm{mC}_y^A(13,\lambda)_{|\mathbf{e}_{23}} &= 0\,,\\
\mathrm{mC}_y^A(13,\lambda)_{|\mathbf{e}_{14}} &= 0\,,\\
\mathrm{mC}_y^A(13,\lambda)_{|\mathbf{e}_{13}} &= (1-s^2)(1-t^2)(1-st)\,.
\end{aligned}$$

For the point $\mathbf{e}_{14}$ we have

$$\begin{aligned}
\mathrm{mC}_y^A(14,\lambda)_{|\mathbf{e}_{24}} &= 0\,,\\
\mathrm{mC}_y^A(14,\lambda)_{|\mathbf{e}_{23}} &= 0\,,\\
\mathrm{mC}_y^A(14,\lambda)_{|\mathbf{e}_{14}} &= (1-t^2)\left(1-\frac{t}{s}\right)\left(1+y\frac{1}{s^2}\right)\,,\\
\mathrm{mC}_y^A(14,\lambda)_{|\mathbf{e}_{13}} &= (1-t^2)(1-st)(1+y)s^{2-2\lceil\lambda\rceil}\,.
\end{aligned}$$

For the point $\mathbf{e}_{23}$ we have

$$\begin{aligned}
\mathrm{mC}_y^A(23,\lambda)_{|\mathbf{e}_{24}} &= 0\,,\\
\mathrm{mC}_y^A(23,\lambda)_{|\mathbf{e}_{23}} &= (1-s^2)\left(1+y\frac{1}{t^2}\right)\left(1+y\frac{s}{t}\right)\,,\\
\mathrm{mC}_y^A(23,\lambda)_{|\mathbf{e}_{14}} &= (1-t^2)\left(1+y\frac{1}{s^2}\right)(1+y)\left(\frac{s}{t}\right)^{\lceil 2\lambda\rceil-1}\,,\\
\mathrm{mC}_y^A(23,\lambda)_{|\mathbf{e}_{13}} &= \begin{cases}(1-st)(1+y)t^{1-\lceil 2\lambda\rceil}(s(1+y)+t+sty) \text{ for } \alpha=0\,,\\ (1-st)(1+y)t^{1-\lceil 2\lambda\rceil}(1+st(1+y)+s^2y) \text{ for } \alpha=1\,.\end{cases}
\end{aligned}$$

For the point $\mathbf{e}_{24}$ we have

$$\begin{aligned}
\mathrm{mC}_y^A(24,\lambda)_{|\mathbf{e}_{24}} &= \left(1+y\frac{1}{t^2}\right)\left(1+y\frac{1}{s^2}\right)\left(1+y\frac{1}{st}\right)\,,\\
\mathrm{mC}_y^A(24,\lambda)_{|\mathbf{e}_{23}} &= \left(1+y\frac{1}{t^2}\right)\left(1+y\frac{s}{t}\right)(1+y)s^{2-2\lceil\lambda\rceil}\,,\\
\mathrm{mC}_y^A(24,\lambda)_{|\mathbf{e}_{14}} &= \left(1+y\frac{1}{s^2}\right)\left(1+y\frac{t}{s}\right)(1+y)t^{2-2\lceil\lambda\rceil}\,,\\
\mathrm{mC}_y^A(24,\lambda)_{|\mathbf{e}_{13}} &= \begin{cases}(1+y)(st)^{1-\lceil 2\lambda\rceil}(st(1+y)-y+s^2y+t^2y+s^2t^2y^2) \text{ for } \alpha=0\,,\\ (1+y)(st)^{1-\lceil 2\lambda\rceil}(st(y+y^2)+1+s^2y+t^2y-s^2t^2y) \text{ for } \alpha=1\,.\end{cases}
\end{aligned}$$

6.2.2. Computation for the variety LG_{23}. The pair $(\mathrm{LG}_{23},\partial\,\mathrm{LG}(23))$ is not SNC, because the variety LG_{23} is not smooth. We take

$$p\colon (Z_{23}, Z_{14}\cup E')\to(\mathrm{LG}_{23},\partial LG(23))$$

as a SNC resolution of singularities. In the previous section we proved that

$$p^*(\mathcal{O}_{\mathrm{LG}(2,4)}(\mathrm{LG}_{23})_{|\,\mathrm{LG}_{23}})\simeq\mathcal{O}_{Z_{23}}(2Z_{14}+E')\,.$$

Therefore

$$p^*\Delta_\lambda = 2\lambda\cdot Z_{14}+\lambda\cdot E'\,.$$

We want to compute the class

$$\begin{aligned}\mathrm{mC}^A_y(23,\lambda) &= \iota_{23*}p_*\,\mathrm{mC}^A_y(Z_{23},Z_{14}\cup E';2\lambda Z_{14}+\lambda E')\\ &= (\iota_{23}\circ p)_*\left(\mathrm{mC}^A_y(Z^o_{23}\subset Z_{23})\cdot\mathcal{O}_{Z_{23}}(\lceil 2\lambda\rceil Z_{14})\cdot\mathcal{O}_{Z_{23}}(\lceil\lambda\rceil E')\right)\,.\end{aligned}$$

Let $\alpha = 2\lceil\lambda\rceil-\lceil 2\lambda\rceil$. Note that $\alpha=1$ when $\lceil\lambda\rceil-\lambda\geq 1/2$ and $\alpha=0$ in the other case. We compute restrictions at the fixed point set.

	$\mathcal{O}_{Z_{23}}(\lceil 2\lambda\rceil Z_{14})$	$\mathcal{O}_{Z_{23}}(\lceil\lambda\rceil E')$	$\mathrm{mC}^A_y(Z^o_{23}\subset Z_{23})$	$\mathrm{mC}^A_y(Z_{23},Z_{14}\cup E';p^*\Delta_\lambda)$
$\mathbf{e}_{23}$	1	1	$(1+y\frac{1}{t^2})(1+y\frac{s}{t})$	$(1+y\frac{1}{t^2})(1+y\frac{s}{t})$
$\mathbf{e}_{14}$	$(s/t)^{\lceil 2\lambda\rceil}$	1	$(1+y\frac{1}{s^2})(1+y)\frac{t}{s}$	$(1+y\frac{1}{s^2})(1+y)(\frac{s}{t})^{\lceil 2\lambda\rceil-1}$
$\mathbf{e}_a$	1	$1/t^{2\lceil\lambda\rceil}$	$(1+y\frac{s}{t})(1+y)t^2$	$(1+y\frac{s}{t})(1+y)t^{1-\lceil 2\lambda\rceil}t^{1-\alpha}$
$\mathbf{e}_b$	$(s/t)^{\lceil 2\lambda\rceil}$	$1/s^{2\lceil\lambda\rceil}$	$(1+y)^2\cdot\frac{t}{s}\cdot s^2$	$(1+y)^2t^{1-\lceil 2\lambda\rceil}s^{1-\alpha}$

Using LRR formula (Theorem 1.24) we compute the pushforward $(\iota_{23}\circ p)_*$ and obtain

$$\mathrm{mC}^A_y(23,\lambda)_{|\mathbf{e}_{24}}=0\,,$$

$$\mathrm{mC}^A_y(23,\lambda)_{|\mathbf{e}_{23}}=(1-s^2)\,\mathrm{mC}^A_y(Z_{23},\partial Z_{23};p^*\Delta_\lambda)_{|\mathbf{e}_{23}}=(1-s^2)\left(1+y\frac{1}{t^2}\right)\left(1+y\frac{s}{t}\right)\,,$$

$$\mathrm{mC}^A_y(23,\lambda)_{|\mathbf{e}_{14}}=(1-t^2)\,\mathrm{mC}^A_y(Z_{14},\partial Z_{23};p^*\Delta_\lambda)_{|\mathbf{e}_{14}}=(1-t^2)\left(1+y\frac{1}{s^2}\right)(1+y)\left(\frac{s}{t}\right)^{\lceil 2\lambda\rceil-1}\,,$$

$$\begin{aligned}\mathrm{mC}^A_y(23,\lambda)_{|\mathbf{e}_{13}} &= eu(T_{\mathbf{e}_{13}}\,\mathrm{LG}(2,4))\cdot\sum_{i\in\{a,b\}}\frac{\mathrm{mC}^A_y(Z_{23},Z_{14}\cup E\;p^*\Delta_\lambda)_{|\mathbf{e}_i}}{eu(T_{\mathbf{e}_i}Z_{23})}\\ &= (1-st)(1+y)t^{1-\lceil 2\lambda\rceil}\left(\frac{(1+y\frac{s}{t})(1-s^2)t^{1-\alpha}}{1-s/t}+\frac{(1+y)(1-t^2)s^{1-\alpha}}{1-t/s}\right)\,.\end{aligned}$$

After calculations we get the formula

$$\mathrm{mC}^A_y(23,\lambda)_{|\mathbf{e}_{13}}=\begin{cases}(1-st)(1+y)t^{1-\lceil 2\lambda\rceil}(s(1+y)+t+sty) & \text{for } \alpha=0\,,\\ (1-st)(1+y)t^{1-\lceil 2\lambda\rceil}(1+st(1+y)+s^2y) & \text{for } \alpha=1\,.\end{cases}$$

6.3. Axioms

In this section we manually check that the computed classes satisfy the axioms of stable envelope. We omit the normalization axiom because its verification is analogous to the proof of proposition 2.5.

6.3.1. Support axiom. According to proposition 5.2 the support axiom states that

- $\mathrm{mC}^A_y(ij,\lambda)_{|\mathbf{e}_{kl}}=0$ if $\mathbf{e}_{ij}<\mathbf{e}_{kl}$.
- For all ij the class $\mathrm{mC}^A_y(ij,\lambda)_{|\mathbf{e}_{24}}$ is divisible by

$$\left(1+y\frac{1}{t^2}\right)\left(1+y\frac{1}{s^2}\right)\left(1+y\frac{1}{st}\right)\,.$$

- For all ij the class $\mathrm{mC}^A_y(ij,\lambda)_{|\mathbf{e}_{23}}$ is divisible by

$$\left(1+y\frac{1}{t^2}\right)\left(1+y\frac{s}{t}\right)\,.$$

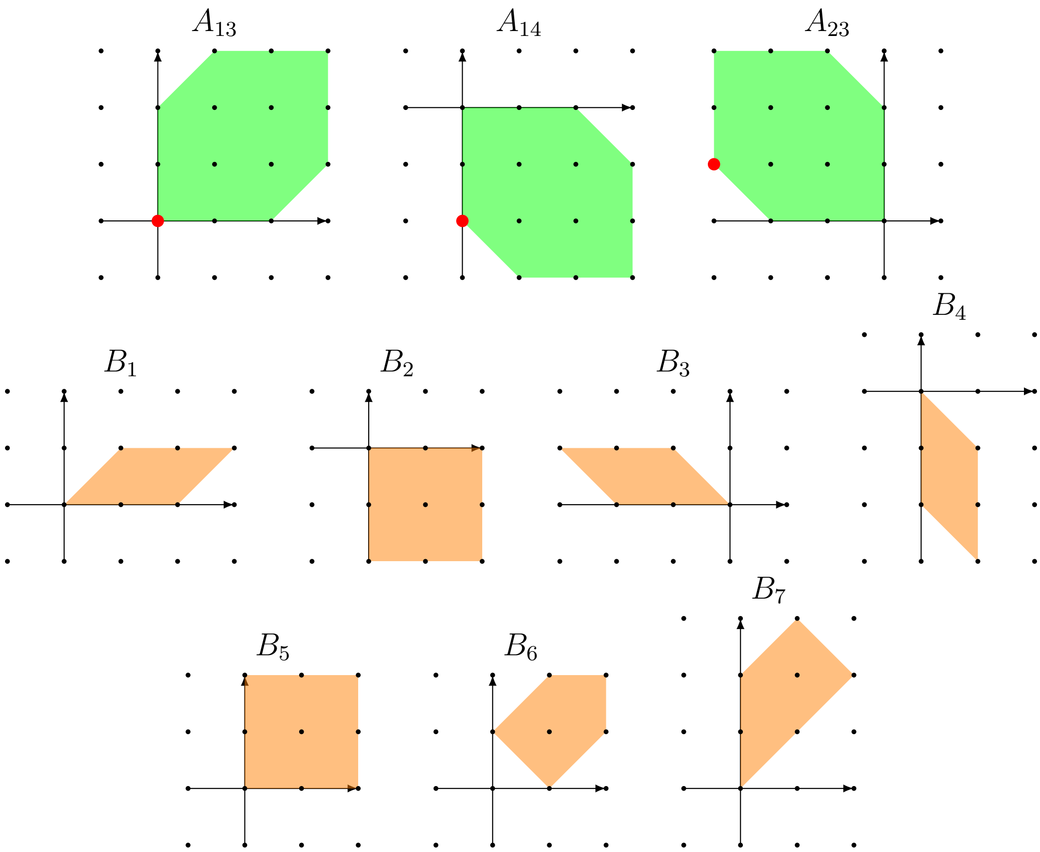
- For all ij the class $\mathrm{mC}_y^A(ij,\lambda)_{|\mathbf{e}_{14}}$ is divisible by

$$\left(1+y\frac{1}{s^2}\right).$$

- For all ij the class $\mathrm{mC}_y^A(ij,\lambda)_{|\mathbf{e}_{13}}$ is divisible by 1.

It is straightforward to check that these conditions hold.

6.3.2. Newton inclusion property. Consider the following convex polygons.



Identify the vector space $\mathbb{R}^2$ with $\mathrm{Hom}(A,\mathbb{C}^*)\otimes_{\mathbb{Z}}\mathbb{R}$ in such a way that the character t corresponds to the vector $(1,0)$ and the character s to $(0,1)$. It is easy to check that

$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(13,\lambda)_{|\mathbf{e}_{13}})=A_{13}\,,\quad \mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(14,\lambda)_{|\mathbf{e}_{14}})=A_{14}\,,\quad \mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(23,\lambda)_{|\mathbf{e}_{23}})=A_{23}\,.$$

Moreover,

$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(14,\lambda)_{|\mathbf{e}_{13}})=B_1+(0,2-2\lceil\lambda\rceil)\,,$$
$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(23,\lambda)_{|\mathbf{e}_{14}})=B_2+(1-\lceil 2\lambda\rceil,\lceil 2\lambda\rceil-1)\,,$$
$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(24,\lambda)_{|\mathbf{e}_{23}})=B_3+(0,2-2\lceil\lambda\rceil)\,,$$
$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(24,\lambda)_{|\mathbf{e}_{14}})=B_4+(2-2\lceil\lambda\rceil,0)\,,$$
$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(24,\lambda)_{|\mathbf{e}_{13}})=B_5+(1-\lceil 2\lambda\rceil,1-\lceil 2\lambda\rceil)\,.$$

For $\lceil\lambda\rceil-\lambda<1/2$ (i.e. for $\alpha=0$) we have

$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(23,\lambda)_{|\mathbf{e}_{13}})=B_6+(1-\lceil 2\lambda\rceil,0)\,.$$

For $\lceil\lambda\rceil - \lambda \geq 1/2$ (i.e. for $\alpha = 1$) we have

$$\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(23,\lambda)_{|\mathbf{e}_{13}}) = B_7 + (1 - \lceil 2\lambda\rceil, 0)\,.$$

Consider the function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = x - \lceil x\rceil\,.$$

The image of g is the interval $(-1, 0]$. The Newton inclusion property is equivalent to the following list of conditions.

- For $\mathbf{e}_{14} > \mathbf{e}_{13}$ we get
$$B_1 + (2 + 2g(\lambda)) \cdot (0,1) \subset A_{13}\,.$$
- For $\mathbf{e}_{23} > \mathbf{e}_{14}$ we get
$$B_2 + (1 + g(2\lambda)) \cdot (1,-1) \subset A_{14}\,.$$
- For $\mathbf{e}_{24} > \mathbf{e}_{23}$ we get
$$B_3 + (2 + 2g(\lambda)) \cdot (0,1) \subset A_{23}\,.$$
- For $\mathbf{e}_{24} > \mathbf{e}_{14}$ we get
$$B_4 + (2 + 2g(\lambda)) \cdot (1,0) \subset A_{14}\,.$$
- For $\mathbf{e}_{24} > \mathbf{e}_{13}$ we get
$$B_5 + (1 + g(2\lambda)) \cdot (1,1) \subset A_{13}\,.$$
- For $\mathbf{e}_{23} > \mathbf{e}_{13}$ and $\lceil\lambda\rceil - \lambda < 1/2$ we get
$$B_6 + (1 + g(2\lambda)) \cdot (1,0) \subset A_{13}\,.$$
- For $\mathbf{e}_{23} > \mathbf{e}_{13}$ and $\lceil\lambda\rceil - \lambda \geq 1/2$ we get
$$B_7 + (1 + g(2\lambda)) \cdot (1,0) \subset A_{13}\,.$$

It can be easily checked that our polytopes satisfy these conditions.
The distinguished point in $\mathcal{N}^{\mathrm{A}}(\mathrm{mC}_y^A(ij,\lambda))_{|\mathbf{e}_{ij}}$ is equal to

$$x_{ij} = \{-\det T^{+}_{\mathbf{e}_{ij}}\, \mathrm{LG}(2,4)\}\,.$$

It is the red point in polytopes A_{13}, A_{14} and A_{23}. The distinguished point axiom says that we may take $A_{ij} \setminus x_{ij}$ in the above containments.

CHAPTER 7

Appendix A: Stable envelopes

7.1. Setting

The notion of the stable envelope is still evolving. Initially, it was defined for symplectic resolutions [**MO19, Oko17, OS16, AO21**]. Later, the definition was generalized to symplectic manifolds, whose BB-cells satisfy certain regularity conditions. See a recent paper [**Oko21**] for a new development in which manifold is not even assumed symplectic.

In this section we will present a version of axioms of the stable envelope. We assume that X is a symplectic manifold, whose BB-cells satisfy certain regularity conditions. We will prove that these axioms agree with [**Oko17, OS16**] for a general enough slope and define a unique element for an arbitrary slope. We do not prove that an element satisfying these axioms exists.

Let (X, ω) be a symplectic $\mathbb{T} = A \times \mathbb{C}^*$-variety. Let $\mathfrak{h}$ be a character of $\mathbb{T}$ equal to the projection to the factor $\mathbb{C}^*$. Suppose that:

- The fixed point set X^A is finite.
- The symplectic form ω is an eigenvector of the torus $\mathbb{T}$ and $\mathfrak{h}$ is its character.

The first condition implies that $X^{\mathbb{T}} = X^A$. The second implies that the torus A preserves the symplectic form ω.

Remark 7.1. Suppose that $\dim X > 0$. Let $\sigma\colon \mathbb{C}^* \to A$ be a one parameter subgroup such that $X^{\mathbb{C}^*} = X^A$. Then, the above conditions imply that there is no open BB-cell in X. Therefore, the variety X cannot be projective.

Definition 7.2 (cf. definition 1.64 and remark 1.65). Suppose that a one parameter subgroup $\sigma : \mathbb{C}^* \to A$ is good for X, i.e. that $X^A = X^{\mathbb{C}^*}$. We say that the pair (X, σ) is admisible if the sum of BB cells $\bigsqcup X^+_{\mathbf{e}}$ is closed in X.

Remark 7.3 ([**MO19**, Lemma 3.2.7]). Suppose that X is a symplectic resolution and the one parameter subgroup σ is good. Then, the pair (X, σ) is admisible.

Definition 7.4. A polarization is an element $T^{1/2} \in K^{\mathbb{T}}(X)$ such that

$$T^{1/2} + \mathbb{C}_{-\mathfrak{h}} \otimes \left(T^{1/2}\right)^* = TX \in K^{\mathbb{T}}(X)\,.$$

Definition 7.5. For a cocharacter σ we denote by $\mathfrak{C}_\sigma$ its weight chamber.

Definition 7.6. Let X be a $\mathbb{T}$-equivariant symplectic variety satisfying the assumptions from the beginning of this section. Let $\sigma\colon \mathbb{C}^* \to A$ be a one parameter subgroup such that the pair (X, σ) is admisible. Let $T^{1/2}$ be a polarization. Consider a fractional line bundle $s \in Pic(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The K-theoretic stable envelope is a set of elements $Stab^s_{\mathfrak{C}_\sigma, T_{1/2}} \in K^{\mathbb{T}}(X)(\mathbf{e})$ indexed by the fixed point set X^A, such that

1. Support axiom: For any fixed point $\mathbf{e} \in X^A$

$$\operatorname{supp}(Stab^s_{\mathfrak{C}_\sigma, T_{1/2}}(\mathbf{e})) \subset \bigsqcup_{\mathbf{e}' \leq \mathbf{e}} X^+_{\mathbf{e}'}\,.$$

2. Normalization axiom: For any fixed point $\mathbf{e} \in X^A$

$$Stab^s_{\mathfrak{C}_\sigma, T_{1/2}}(\mathbf{e})_{|\mathbf{e}} = eu(T^-_{\mathbf{e}} X) \frac{(-1)^{\operatorname{rk} T^{1/2,\,\geq 0}_{|\mathbf{e}}}}{\det T^{1/2,\,\geq 0}_{|\mathbf{e}}}\,.$$

3. Newton inclusion property: Choose any A-linearisation of the slope s. For a pair of fixed points $\mathbf{e}', \mathbf{e} \in M^A$ such that $\mathbf{e}' \leq \mathbf{e}$ we have a containment of the Newton polytopes

$$\mathcal{N}^{\mathrm{A}}\left(Stab^s_{\mathfrak{C}_\sigma, T_{1/2}}(\mathbf{e})_{|\mathbf{e}'}\right) + \mathrm{w}_{\mathbf{e}}(s) \subset \mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}'} X)) - \mathrm{w}_{\mathbf{e}'}\left(\det T^{1/2,\,\geq 0}_{|\mathbf{e}'}\right) + \mathrm{w}_{\mathbf{e}'}(s)\,.$$

4. Distinguished point: Choose any A-linearisation of the slope s. For a pair of fixed points $\mathbf{e}', \mathbf{e} \in M^A$ such that $\mathbf{e}' < \mathbf{e}$ the point

$$\mathrm{w}_{\mathbf{e}'}(s) - \mathrm{w}_{\mathbf{e}}(s) - \mathrm{w}_{\mathbf{e}'}\left(\det T^{1/2,\,\geq 0}_{|\mathbf{e}'}\right) \in \operatorname{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}$$

does not belong to the Newton polytope $\mathcal{N}^{\mathrm{A}}\left(Stab^s_{\mathfrak{C}_\sigma, T_{1/2}}(\mathbf{e})_{|\mathbf{e}'}\right)$.

Remark 7.7. In the above definition $T^{1/2,\,\geq 0}_{|\mathbf{e}'}$ denotes the nonnegative part of the virtual bundle $T^{1/2}_{|\mathbf{e}'} \in K^{\mathbb{T}}(\mathbf{e}')$ (see proposition 1.13).

Remark 7.8. Suppose that $X = T^*M$ is a cotangent variety with the $\mathbb{T}$-action described in example 1.61. Choose $T^{1/2} = \pi^* TM$. Then the above definition is equivalent to definition 1.68.

Remark 7.9 ([**Oko17**, Paragraph 9.1.12]). Stable envelopes corresponding to different polarization are related by a shift of slope and renormalization. Therefore, theorem 5.12 may be used to obtain a formula for the stable envelope for an arbitrary polarization.

7.2. Comparison of axioms

Proposition 7.10. *Suppose that the slope s is general enough, i.e. that*

$$\mathrm{w}_{\mathbf{e}'}(s) - \mathrm{w}_{\mathbf{e}}(s) - \mathrm{w}_{\mathbf{e}'}\left(\det T^{1/2,\,\geq 0}_{|\mathbf{e}'}\right)$$

is not a lattice point. Then definition 7.6 is equivalent to the one from [**Oko17**, Chapter 9] *and* [**OS16**, Section 2.1] *i.e.*

$$Stab^s_{\mathfrak{C}_\sigma, T_{1/2}}(\mathbf{e}) = \mathfrak{h}^{-(1/2)\operatorname{rk} T^{1/2,\,\geq 0}_{|\mathbf{e}}} \cdot \widehat{Stab}^s_{\mathfrak{C}_\sigma, T_{1/2}}(1_{\mathbf{e}})$$

where $\widehat{Stab}^s_{\mathfrak{C}_\sigma, T_{1/2}}$ is the stable envelope morphism from Okounkov's papers.

Proof. Note that for a general enough slope the fourth axiom (distinguished point) follows from the third one (Newton inclusion property). The considered point is a vertex of the polytope

$$\mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}'} X)) - \mathrm{w}_{\mathbf{e}'}\left(\det T^{1/2,\,\geq 0}_{|\mathbf{e}'}\right) + \mathrm{w}_{\mathbf{e}'}(s) - \mathrm{w}_{\mathbf{e}}(s)\,,$$

which is not a lattice point. Therefore, it cannot lie in the smaller lattice polytope

$$\mathcal{N}^{\mathrm{A}}\left(Stab^{s}_{\mathfrak{C}_\sigma,T_{1/2}}(\mathbf{e})_{|\mathbf{e}'}\right).$$

Therefore, it is enough to show that the first three axioms are equivalent to Okounkov's axioms.

According to [**Oko17**, Chapter 9] and [**OS16**, Section 2.1] the stable envelope is a map of $K^{\mathbb{T}}(pt)$-modules

$$K^{\mathbb{T}}(X^A)\to K^{\mathbb{T}}(X)$$

given by a correspondence

$$\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}\in K^{\mathbb{T}}(X^A\times X),$$

which satisfies three properties [**Oko17**, Paragraph 9.1.3]. Below we denote both morphism and correspondence by $\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}$.

The set X^A is finite, therefore the morphism $\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}$ is uniquely determined by a set of elements

$$\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}(\mathbf{e}):=\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}(1_{\mathbf{e}}).$$

The main ingredient in Okounkov's definition are attracting sets (cf. [**OS16**, Paragraph 2.1.3], [**Oko17**, Paragraph 9.1.2]). The straightforward comparison of definitions shows that they coincide with the BB-cells

$$Attr(\mathbf{e})=\{x\in X\,|\,\lim_{t\to 0}\sigma(t)\cdot x=\mathbf{e}\}=X^+_{\mathbf{e}}.$$

Support axiom: ([**OS16**, Paragraph 2.1.1], [**Oko17**, Paragraph 9.1.3 (1)], or [**MO19**, Theorem 3.3.4 (i)]) It is required that

$$\operatorname{supp}(\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}})\subset\bigsqcup_{\mathbf{e}\in X^A}\left(\mathbf{e}\times\bigsqcup_{\mathbf{e}'\le\mathbf{e}}Attr(\mathbf{e}')\right).$$

This means that for any fixed point $\mathbf{e}\in X^A$

$$\operatorname{supp}(\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}(\mathbf{e}))\subset\bigsqcup_{\mathbf{e}'\le\mathbf{e}}Attr(\mathbf{e}').$$

The attracting sets coincide with the BB-cells, so this is an equivalent formulation of the first axiom from definition 7.6.

Normalization axiom: ([**OS16**, Paragraph 2.1.4], [**Oko17**, Paragraph 9.1.5]) For any fixed point $\mathbf{e}\in X^A$. the correspondence $\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}$ satisfies

$$\left(\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}\right)_{|\mathbf{e}\times\mathbf{e}}=(-1)^{\operatorname{rk}T^{1/2,\,\ge0}_{|\mathbf{e}}}\left(\frac{\det\nu^-(\mathbf{e}\subset X)}{\det T^{1/2,\,\neq0}_{|\mathbf{e}}}\right)^{1/2}\otimes\mathcal{O}_{Attr}|_{\mathbf{e}\times\mathbf{e}},\tag{19}$$

where

$$Attr:=\{(y,x)\in X^A\times X\,|\,\lim_{t\to0}\sigma(t)\cdot x=y\}.$$

By definition

$$\left(\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}\right)_{|\mathbf{e}\times\mathbf{e}}=\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}(\mathbf{e}).$$

In [**OS16**, Paragraph 2.1.4] it is noted that

$$\begin{aligned} \mathcal{O}_{Attr}|_{\mathbf{e}\times\mathbf{e}} &= \mathcal{O}_{\mathrm{diag}\,\mathbf{e}} \cdot eu(\nu^-(\mathbf{e}\subset X)) = eu(T^-_{\mathbf{e}}X)\,, \\ \frac{\det \nu^-(\mathbf{e}\subset X)}{\det T^{1/2,\neq 0}_{|\mathbf{e}}} &= \frac{\mathfrak{h}^{\mathrm{rk}\, T^{1/2,\,\geq 0}_{|\mathbf{e}}}}{\left(\det T^{1/2,\,\geq 0}_{|\mathbf{e}}\right)^2}\,. \end{aligned}$$

The equation 19 simplifies to

$$\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}(\mathbf{e}) = \mathfrak{h}^{(1/2)\,\mathrm{rk}\, T^{1/2,\,\geq 0}_{|\mathbf{e}}} \cdot eu(T^-_{\mathbf{e}}X)\cdot \frac{(-1)^{\mathrm{rk}\, T^{1/2,\,\geq 0}_{|\mathbf{e}}}}{\det T^{1/2,\,\geq 0}_{|\mathbf{e}}}$$

Up to normalization by $\mathfrak{h}^{(1/2)\,\mathrm{rk}\, T^{1/2,\,\geq 0}_{|\mathbf{e}}}$ this is the second axiom from definition 7.6.

Newton inclusion property: ([**OS16**, Paragraph 2.1.6], [**Oko17**, Paragraph 9.1.9]) In the case of isolated fixed points, the last axiom of stable envelope $\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}$ states that for any pair of fixed points $\mathbf{e},\mathbf{e}'\in X^A$ we have

$$\mathcal{N}^{\mathrm{A}}\left(\left(\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}\right)_{|\mathbf{e}\times\mathbf{e}'}\cdot s_{|\mathbf{e}}\right) \subset \mathcal{N}^{\mathrm{A}}\left(\left(\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}\right)_{|\mathbf{e}'\times\mathbf{e}'}\cdot s_{|\mathbf{e}'}\right)\,.$$

The support condition implies that this requirement is nontrivial only when $\mathbf{e}\geq\mathbf{e}'$. The normalization axiom implies that the above formula is equivalent to

$$\mathcal{N}^{\mathrm{A}}\left(\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}(\mathbf{e})_{|\mathbf{e}'}\cdot s_{|\mathbf{e}}\right) = \mathcal{N}^{\mathrm{A}}\left(\mathfrak{h}^{(1/2)\,\mathrm{rk}\, T^{1/2,\,\geq 0}_{|\mathbf{e}'}} \cdot eu(T^-_{\mathbf{e}'}X)\cdot \frac{(-1)^{\mathrm{rk}\, T^{1/2,\,\geq 0}_{|\mathbf{e}'}}}{\det T^{1/2,\,\geq 0}_{|\mathbf{e}'}}\cdot s_{\mathbf{e}'}\right)\,.$$

We can simplify the above formula

$$\mathcal{N}^{\mathrm{A}}\left(\widehat{Stab}^{s}_{\mathfrak{C}_\sigma,T_{1/2}}(\mathbf{e})_{|\mathbf{e}'}\right) + \mathrm{w}_{\mathbf{e}}(s) \subset \mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}'}X)) - \mathrm{w}_{\mathbf{e}'}\left(\det T^{1/2,\,\geq 0}_{|\mathbf{e}'}\right) + \mathrm{w}_{\mathbf{e}'}(s)\,.$$

This is exactly the third axiom from definition 7.6. □

7.3. Uniqueness

For a slope which is not generic, the first three axioms of the stable envelope do not define an unique class (see the example below). In this section we prove that addition of the forth axiom resolves this problem. For a general enough slope uniqueness of the stable envelope was proved in [**Oko17**, Proposition 9.2.2]. For the sake of completeness, we present here the proof for an arbitrary slope. The proof is a generalisation of the proof of uniqueness of cohomological envelopes [**MO19**, Paragraph 3.3.4].

Example 7.11. Consider a variety $M=\mathbb{P}^1$ with the $A=\mathbb{C}^*$-action given by

$$t\cdot[a:b] = [ta:b]\,.$$

Consider the cotangent variety $X = T^*\mathbb{P}^1$ with the $\mathbb{T}$-action described in example 1.61. Denote by α the character of $\mathbb{T}$ corresponding to the projection to A. The pair (X, id_A) is admisible.

Consider the stable envelope for the one parameter subgroup id_A, the tangent bundle $T\mathbb{P}^1$ as a polarization and the trivial line bundle θ as a slope. Both

$$Stab^{\theta}([1:0]) = 1 - O(-1),\ Stab^{\theta}([0:1]) = \frac{1}{\mathfrak{h}} - \frac{O(-1)}{\alpha}$$

and

$$Stab^{\theta}([1:0]) = 1 - O(-1),\ Stab^{\theta}([0:1]) = \frac{O(-1)}{\mathfrak{h}} - \frac{O(-2)}{\alpha}$$

satisfy the first three axioms of the stable envelope.

Proposition 7.12. *Consider the situation as in definition 7.6. There exists at most one class satisfying the axioms of the stable envelope.*

Lemma 7.13. *Choose a set of fractional weights $w_{\mathbf{e}} \in \mathrm{Hom}(A, \mathbb{C}^*) \otimes \mathbb{Q}$ indexed by the fixed point set X^A. Suppose that an element $a \in K^{\mathbb{T}}(X)$ satisfies two conditions*

(1) $\mathrm{supp}(a) \subset \bigsqcup_{\mathbf{e} \in X^A} X^+_{\mathbf{e}}$,

(2) For any fixed point $\mathbf{e} \in X^A$ we have a containment of Newton polytopes

$$\mathcal{N}^{\mathrm{A}}(a_{|\mathbf{e}}) \subset \big(\mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}}X)) \setminus \{0\}\big) + w_{\mathbf{e}} \,.$$

Then $a = 0$.

Proof. We proceed by induction on the partially ordered set X^A. Suppose that the element a is supported on the closed set $Y = \bigsqcup_{\mathbf{e} \in Z} X^+_{\mathbf{e}}$ for some subset $Z \subset X^A$. There is a fixed point $\mathbf{e}_1 \in X^A$, such that the BB cell $X^+_{\mathbf{e}_1}$ is an open subvariety in Y. We aim to show that a is supported on the closed subset $\bigsqcup_{\mathbf{e} \in Z \setminus \{\mathbf{e}_1\}} X^+_{\mathbf{e}}$. By induction it implies that $a = 0$.

Choose an open subset $U \subset X$ such that $U \cap Y = X^+_{\mathbf{e}_1}$. Then $X^+_{\mathbf{e}_1} \subset U$ is an inclusion of a smooth, closed subvariety. Consider a commutative diagram

$$\begin{array}{ccccc} & & U & \xrightarrow{i} & X \\ & & \uparrow{\scriptstyle \tilde{j}} & & \uparrow{\scriptstyle j} \\ \mathbf{e}_1 & \xrightarrow{s_0} & X^+_{\mathbf{e}_1} & \xrightarrow{\tilde{i}} & Y \end{array}$$

The square in the diagram is a pullback. The element a is supported on Y, so there exists an element $\alpha \in G^{\mathbb{T}}(Y)$ such that $j_*(\alpha) = a$. Then

$$a_{|\mathbf{e}_1} = (j_*\alpha)_{|\mathbf{e}_1} = s_0^*\tilde{j}^* i^* j_* \alpha = s_0^*\tilde{j}^*\tilde{j}_*\tilde{i}^*\alpha = eu(X^+_{\mathbf{e}_1} \subset U) \cdot \alpha_{|\mathbf{e}_1} = eu(T^-_{\mathbf{e}_1}X) \cdot \alpha_{|\mathbf{e}_1} \,.$$

The third equality follows form proposition 1.9 (1) and the fourth from proposition 1.18. It follows that:

$$\mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}_1}X) \cdot \alpha_{|\mathbf{e}_1}) = \mathcal{N}^{\mathrm{A}}(a_{|\mathbf{e}_1}) \subset \big(\mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}_1}X)) \setminus \{0\}\big) + w_{\mathbf{e}_1}. \tag{20}$$

Assume that $\alpha_{|\mathbf{e}_1}$ is a nonzero element. Then the Newton polytope $\mathcal{N}^{\mathrm{A}}(\alpha_{|\mathbf{e}_1})$ is nonempty. The ring $K^{\mathbb{C}^*}(\mathbf{e}_1)$ is a domain so proposition 1.28 (b) implies that

$$\mathcal{N}^{\mathrm{A}}\left(eu(T^-_{\mathbf{e}_1}X)\right) \subset \mathcal{N}^{\mathrm{A}}\left(eu(T^-_{\mathbf{e}_1}X)\right) + \mathcal{N}^{\mathrm{A}}(\alpha_{|\mathbf{e}_1}) = \mathcal{N}^{\mathrm{A}}\left(eu(T^-_{\mathbf{e}_1}X) \cdot \alpha_{|\mathbf{e}_1}\right). \tag{21}$$

Inclusions (20) and (21) imply that

$$\mathcal{N}^{\mathrm{A}}\left(eu(T^-_{\mathbf{e}_1}X)\right) \subset \big(\mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}_1}X)) \setminus \{0\}\big) + w_{\mathbf{e}_1} \,.$$

But no polytope can be translated into a proper subset of itself. This contradiction proves that the element $\alpha_{|\mathbf{e}_1}$ is equal to zero. The map s_0 is a section of an affine bundle so it induces an isomorphism of the K-theory. It follows that $\alpha_{|X^+_{\mathbf{e}_1}} = 0$. Thus, the element α is supported on the closed set $\bigsqcup_{\mathbf{e}\in Z\setminus\{\mathbf{e}_1\}} X^+_{\mathbf{e}}$. Therefore, a is also supported on this set. □

Proof of proposition 7.12. Let $\{Stab(\mathbf{e})\}_{\mathbf{e}\in X^A}$ and $\{\widetilde{Stab}(\mathbf{e})\}_{\mathbf{e}\in X^A}$ be two sets of elements satisfying the axioms of the stable envelope. It is enough to show that for a given fixed point $\mathbf{e}\in X^A$ the element $Stab(\mathbf{e})-\widetilde{Stab}(\mathbf{e})$ satisfies conditions of lemma 7.13 for some set of vectors $w_{\mathbf{e}}$. Let

$$w_{\mathbf{e}'} = \mathrm{w}_{\mathbf{e}'}(s) - \mathrm{w}_{\mathbf{e}}(s) - \mathrm{w}_{\mathbf{e}'}\left(\det\left(T^{1/2}_{\mathbf{e}'}\right)^{\geq 0}\right).$$

The first condition of lemma 7.13 follows from the support axiom. Let's focus on the second condition. The only nontrivial case is $\mathbf{e}' < \mathbf{e}$. In the other cases we have

$$Stab(\mathbf{e})_{|\mathbf{e}'} - \widetilde{Stab}(\mathbf{e})_{|\mathbf{e}'} = 0.$$

When $\mathbf{e}' < \mathbf{e}$ the last two axioms imply that the Newton polytopes $\mathcal{N}^{\mathrm{A}}(Stab(\mathbf{e})_{|\mathbf{e}'})$ and $\mathcal{N}^{\mathrm{A}}(\widetilde{Stab}(\mathbf{e})_{|\mathbf{e}'})$ are contained in a convex set

$$\big(\mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}'}X)) \setminus \{0\}\big) + w_{\mathbf{e}'}.$$

Thus

$$\begin{aligned}\mathcal{N}^{\mathrm{A}}\left(Stab(\mathbf{e})_{|\mathbf{e}'} - \widetilde{Stab}(\mathbf{e})_{|\mathbf{e}'}\right) &\subset \mathrm{conv}\left(\mathcal{N}^{\mathrm{A}}(Stab(\mathbf{e})_{|\mathbf{e}'}), \mathcal{N}^{\mathrm{A}}(\widetilde{Stab}(\mathbf{e})_{|\mathbf{e}'})\right)\\ &\subset \big(\mathcal{N}^{\mathrm{A}}(eu(T^-_{\mathbf{e}'}X)) \setminus \{0\}\big) + w_{\mathbf{e}'}\,.\end{aligned}$$

□

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