Complex Manifolds, due 18.12.2023
Riemann Surfaces (again)

Exercise 1. (a) Prove that every Riemann surface is Kähler.
(b) Using Kodaira embedding theorem, infer that every compact Riemann surface is projective.

Another, more explicit proof of part (b) will be given in exercises below.

Notation. In the following, X is a compact Riemann surface of genus g, and K_X is its canonical divisor. For any divisor D on X, we write $h^i(D) \coloneqq \dim H^i(\mathcal{O}_X(D))$.

Exercise 2. Let D be a divisor on X.
(a) Prove that $\chi(\mathcal{O}_X) = 1 - g$.
(b) Prove the *Riemann–Roch formula*: $\chi(\mathcal{O}_X(D)) = 1 - g + \deg D$.
(*Hint:* use induction on $\deg D$ and an exact sequence $0 \longrightarrow \mathcal{O}_X(-p) \longrightarrow \mathcal{O}_X \longrightarrow \mathbb{C}_p$ for a point $p \in X$, where $\mathbb{C}_p$ is the skyscraper sheaf on p)
(c) Using Serre duality, prove that $\chi(\mathcal{O}_X) = h^0(D) - h^0(K_X - D)$.

Exercise 3. Prove (using either Riemann–Roch, or Hurwitz formula) that $\deg K_X = 2g - 2$.

Definition. Let L be a line bundle. We say that L is *globally generated* if the common zero locus of all its nonzero sections is empty; so the map $\varphi_L \colon X \longrightarrow \mathbb{P}(H^0(L))$ is everywhere defined; see Exercise 3 from set 2. We say that L is *very ample* if φ_L is an embedding.

Exercise 4. Let D be a divisor on X. Prove that
(a) $\mathcal{O}_X(D)$ is globally generated if and only if $h^1(D - p) = 0$ for all points $p \in X$ (i.e. $|D|$ *is base point free*)
(b) $\mathcal{O}_X(D)$ is very ample if and only if $h^1(D - p - q) = 0$ for all points $p, q \in X$ (i.e. $|D|$ *separates points and tangent vectors*)
(c) Check directly that the above conditions hold for D being a point on $\mathbb{P}^1$.

Exercise 5. Let D be a divisor on X.
(a) Using Exercise 4(b) prove that the line bundle $\mathcal{O}_X(D)$ is very ample if $\deg D \geqslant 2g + 1$.
(b) Deduce that every compact Riemann surface is projective.
(c) Is the bound in (a) optimal?

Definition. We say that X is *hyperelliptic* if it admits a holomorphic map $X \longrightarrow \mathbb{P}^1$ of degree 2. One can construct hyperelliptic curves explicitly, as follows (cf. Exercise 11 from set 1).

Exercise 6. Fix $g \geqslant 1$. Let $p \in \mathbb{C}[x]$ be a polynomial of degree $2g + 2$, with no multiple roots. Define a compact Riemann surface X by gluing two copies of an affine curve $\{y^2 = p(x)\} \subseteq \mathbb{C}^2$ via

$$(x, y) \mapsto \left(\frac{1}{x}, \frac{y}{x^{g+1}} \right).$$

(a) Prove that the projection $(x, y) \mapsto x$ extends to a 2-1 cover $X \longrightarrow \mathbb{P}^1$, ramified at the roots of p.
(b) Using Hurwitz formula, prove that X has genus g.
(c) Prove that the forms $\frac{x^j}{y}\, dx$ for $j \in \{0, \ldots, g - 1\}$ form a basis of $H^{1,0}(X)$.
(d) Deduce that the map $X \longrightarrow \mathbb{P}^{g-1}$ given by the canonical line bundle is the composition of the projection from (a) and the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$.
(e) What happens if we take a polynomial p of odd degree?
(f) Is X the same as the closure of $\{y^2 = p(x)\}$ in $\mathbb{P}^2$?

Exercise* 7. Give an alternative proof of Exercise 6(b), showing that, as a smooth manifold, X can be constructed as follows. Let $a_1, \ldots, a_{2g+2} \in \mathbb{C}$ be roots of p. Cut slits in the affine plane $\mathbb{C}$ along some paths joining a_1 with a_2, a_3 with a_4, etc., and glue two copies of $\mathbb{C}$ along those slits.

Exercise 8 (Canonical embedding)**.** Assume that $g \geqslant 2$. Using Exercise 4(b), prove that the canonical divisor K_X is very ample if and only if X is not hyperelliptic.

Exercise 9. Deduce from Exercise 8 that if $g \geqslant 2$ then the automorphism group of X is finite.
(*Hint:* if X is not hyperelliptic, embed it into $\mathbb{P}^{g-1}$ using the canonical bundle, and prove that automorphisms of X extend to automorphisms of $\mathbb{P}^{g-1}$).
(*Bonus vague question:* Can you generalize this argument to higher dimensions?)

Exercise* **10** (Hurwitz theorem)**.** Assume that $g \geqslant 2$. Applying Hurwitz formula to the quotient map $X \longrightarrow X/\operatorname{Aut}(X)$, prove that $\#\operatorname{Aut}(X) \leqslant 84(g-1)$.

Exercise 11. (a) Prove that every compact Riemann surface of genus 1 is hyperelliptic.
(b) Prove that a quartic curve in $\mathbb{P}^2$ has genus 3, but is not hyperelliptic.
(c) Prove that every genus 3 curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is hyperelliptic.

Exercise* **12** (The Abel–Jacobi map)**.** Let Y be a compact Kähler manifold.
(a) Prove that the image of the natural map $H^1(Y,\mathbb{Z}) \longrightarrow H^1(Y,\mathcal{O}_Y)$ is a lattice of maximal rank.
(b) Using the exponential sequence, identify the quotient $H^1(Y,\mathcal{O}_Y)/H^1(Y,\mathbb{Z})$ with the group $\operatorname{Pic}^0(Y)$ of line bundles with trivial first Chern class. This group is called the *Picard torus* of Y.
(c) Identify the dual torus to $\operatorname{Pic}^0(Y)$ with $H^0(Y,\Omega_Y)^*/H_1(Y,\mathbb{Z})$, where $\gamma \in H_1(Y,\mathbb{Z})$ corresponds to a functional $\alpha \mapsto \int_\gamma \alpha$. This group is called the *Albanese torus*, and is denoted by $\operatorname{Alb}(Y)$.
(d) Fix a base point $x_0 \in Y$. The *Abel–Jacobi map* $\mu\colon Y \longrightarrow \operatorname{Alb}(Y)$ associates to x a functional $[\alpha \mapsto \int_{x_0}^x \alpha]$, where the integral is taken along any path in X joining x_0 with x. Prove that this map is well defined and holomorphic.
(e) Prove that the Abel–Jacobi map induces a surjection on the level of holomorphic 1-forms.
(f) Prove that if Y is a complex torus $\mathbb{C}^n/\Lambda$ then the Abel–Jacobi map is an isomorphism.
(g) Prove that for $n \gg 0$, the map $Y^n \ni (x_1,\dots,x_n) \mapsto \sum_{i=1}^n \mu(x_i) \in \operatorname{Alb}(Y)$ is surjective, and submersive over a dense open subset in $\operatorname{Alb}(Y)$.

Abel–Jacobi theorem. On a compact Riemann surface X, $\operatorname{Pic}^0(X)$ is naturally identified with $\operatorname{Alb}(X)$, and the Abel–Jacobi map becomes $x \mapsto \mathcal{O}_X(x - x_0)$. Either of these tori is called the *Jacobian* of X, and denoted by $J(X)$.

Exercise 13. Let X, E be hyperelliptic curves given by $y^2 = x^6 + 1$ and $y^2 = x^3 + 1$, see Exercise 6. Consider a map $X \longrightarrow E \times E$ given by $(x,y) \mapsto ((x^2,y),(x^{-2},yx^{-3})$. Prove that the induced map $J(X) \longrightarrow J(E\times E) \cong E \times E$ is a quotient by a finite subgroup.

Exercise 14. Let X be a Riemann surface of genus 1.
(a) Using Abel–Jacobi theorem, prove that X is isomorphic to a complex torus $\mathbb{C}/\Lambda$.
(b) Prove that the series
$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$
is absolutely convergent on $\mathbb{C} \setminus \Lambda$, and defines a meromorphic function on X.
(c) * Prove that the formula $[\wp : \wp' : 1]\colon X \longrightarrow \mathbb{P}^2$ defines an embedding.