Complex manifolds, due 11.12.2023
Calibrated geometry

Definition. Let M be a compact oriented Riemanian manifold. We say that a closed k-form $\alpha \in \Omega^k(M)$ is a *calibration* if for every $x \in M$, and every k-dimensional subspace $W \subseteq T_xM$, we have $\alpha|_W \leqslant \mathrm{vol}_W$, where vol_W is the Riemannian volume form. We say that W is *calibrated* if $\alpha|_W = \mathrm{vol}_W$. A closed k-dimensional submanifold $N \subseteq M$ is *calibrated* if $T_xN \subseteq T_xM$ is calibrated for every $x \in N$.

Exercise 1. Prove that a calibrated submanifold minimizes volume in its homology class.

Exercise 2. Let V be a complex vector space, with a J-compatible form ω and associated metric g. Let $W \subseteq V$ be a real subspace of dimension $2k$.

(a) Prove that W^* admits a g-orthonormal basis $x_1, y_1, \ldots, x_k, y_k$ such that $\omega = \sum_{j=1}^k x_j \wedge y_j$.

(b) (Wirtinger inequality) Let vol_{2k} be the volume form on $\bigwedge^{2k} W$ induced by g. Prove that

$$\tfrac{1}{k!}\omega^k|_W \leqslant \mathrm{vol}_W,$$

with equality if and only if W is a complex subspace, i.e. $JW = W$ (*Hint:* use part (a) to reduce the proof to the case $k = 1$, which boils down to the Cauchy–Schwarz inequality).

Deduce that on a compact Kähler manifold (X, ω), k-dimensional complex submanifolds are calibrated with respect to $\frac{1}{k!}\omega^k$ (in particular, they are minimal).

Exercise 3. Let V, g, ω be as in Exercise 2, let $\Omega \in \bigwedge^{n,0} V^*$ be an n-form such that

$$(*) \qquad \tfrac{1}{n!}\omega^n = (\tfrac{\imath}{2})^n \cdot (-1)^{\frac{1}{2}n(n-1)} \Omega \wedge \overline{\Omega},$$

and let $W \subseteq V$ be an oriented, n-dimensional subspace.

(a) Prove that $\omega|_W = 0$ (i.e. W is *Lagrangian*) if and only if W is orthogonal to JW.

(b) Let $w_1, \ldots, w_n$ be an orthonormal basis of W. Prove that $|\Omega(w_1, \ldots, w_n)| \leqslant 1$, and equality holds if and only if W is Lagrangian. (*Hint:* let e_i, Je_i be a standard basis. consider a linear map $e_i \mapsto w_i$, $Je_i \mapsto Jw_i$, and use Hadamard inequality)

(c) Prove that $\mathrm{Re}\,\Omega|_W \leqslant \mathrm{vol}_W$, with equality if and only if $\omega|_W = 0$ and $\mathrm{Im}\,\Omega|_W = 0$, i.e. if W is *special Lagrangian*.

Deduce that on a Calabi–Yau manifold, i.e. a compact Kähler manifold (X, ω) with a holomorphic n-form Ω satisfying equality $(*)$, special Lagrangian submanifolds are $\mathrm{Re}\,\Omega$ - calibrated (hence minimal).