Complex manifolds, due 27.11.2023
More Kähler forms

Last time, we finished discussing series due 6.11 and 13.11. Next time, we will discuss series due 20.11 and this one (you can declare exercises from both of them).

Exercise 1. Let X be a compact complex manifold. The *Kähler cone* is the subset $\mathcal{K}_X \subseteq H^{1,1}(X) \cap H^2(X,\mathbb{R})$ consisting of classes represented by Kähler forms.

(a) Prove that $\mathcal{K}_X$ is indeed a cone, i.e. for every two Kähler forms ω_1, ω_2 the convex combination $t\omega_1 + (1-t)\omega_2$, $t \in [0,1]$; and a multiple $c\omega_1$ for $c > 0$, are Kähler.
(b) Prove that $\mathcal{K}_X$ is open.
(c) Recall that a form is *symplectic* if it is nondegenerate at every point (so every Kähler form is symplectic). Prove that the space of symplectic forms may not be a cone.
(d) Prove that if X is projective then the Kähler cone meets the lattice $H^2(X;\mathbb{Z})$.
(*Remark:* the Kodaira embedding theorem, which will appear later, asserts that the converse is true as well, i.e. a compact complex manifold with an integral Kähler class is projective.)
(e)* Check directly that in Exercise 7 from the previous series, the Kähler cone does not meet $H^2(X;\mathbb{Z})$.

Exercise 2. Let L be a line bundle on a complex manifold, equipped with a hermitian norm h.

(a) Let s be a non-zero section of L. Prove that $\omega_h := -dd^c \log h(s)$ is a well defined, real $(1,1)$-form, which does not depend on the choice of s. We call it a *curvature form of* (L,h).
(*Remark:* we will see later that ω_h is indeed the curvature form of the unique J-compatible hermitian connection on (L,h), called the *Chern connection*).
(b) Prove that the cohomology class of $[\omega_h]$ depends only on L, not on h.
(c) Assume that L is globally generated by sections $s_0,\dots,s_n$, and let $\varphi\colon X \longrightarrow \mathbb{P}^n$ be the corresponding map. Define a hermitian norm h on L by the formula

$$s \mapsto \frac{|\psi(s)|^2}{\sum_{i=0}^n |\psi(s_i)|^2}$$

where $\psi\colon L|_U \longrightarrow U \times \mathbb{C} \xrightarrow{\mathrm{pr}_{\mathbb{C}}} \mathbb{C}$ is some local trivialization of L. Prove that $\omega_h = \varphi^*\omega_{\mathrm{FS}}$. In particular, if $L = \mathcal{O}_{\mathbb{P}^n}(1)$ then one recovers this way the Fubini–Study form.
(d) Express the form $d^c \log|z|$ in polar coordinates $z = re^{i\theta}$ of $\mathbb{C}^*$.
(e) (This is a vague question) Let D be a smooth hypersurface in X, and let s be a section of $\mathcal{O}_X(D)$ vanishing at D. Using the above local expression for $d^c \log||s||$, interpret the curvature form as "an obstruction to trivializing the normal circle bundle to D"
(f) Compute the curvature forms of L^* and $L \otimes L'$ for another line bundle L'.
(g)* Prove that ω_h represents the first Chern class of L, i.e. the image of L by the connecting homomorphism $\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X^*) \longrightarrow H^2(X,\mathbb{Z})$ in the exponential sequence.

Exercise 3. Let (X,ω_X) be a Kähler manifold, and let $\pi\colon Y \longrightarrow X$ be a blowup at a point with exceptional divisor E. Let ω_E be the curvature form of the line bundle $\mathcal{O}_X(-E)$; equipped with a hermitian norm as in Exercise 2(c). Prove that $\pi^*\omega_X$ is not Kähler, but for any small $\varepsilon > 0$, the sum $\pi^*\omega_X + \varepsilon\omega_E$ is Kähler.

Exercise 4 ($\partial\overline{\partial}$-lemma)**.** Let X be a compact Kähler manifold.

(a) Let α be a d-exact (p,q)-form for some $p,q \geqslant 1$. Prove that α is orthogonal to the space of harmonic forms. Deduce that α is ∂-, $\overline{\partial}$- and $\partial\overline{\partial}$-exact: the last condition means that there is a $(p-1,q-1)$-form β such that $\alpha = \partial\overline{\partial}\beta$.
(b) Conclude that for any two Kähler forms ω, ω' in the same cohomology class, there is a smooth function f, called a *potential*, such that $\omega' = \omega + dd^c f$; cf. Exercise 2(b). Describe potentials comparing Fubini–Study forms on $\mathbb{P}^n$ obtained using different coordinates on $\mathbb{C}^{n+1}$.
(c) Prove that the $\partial\overline{\partial}$-lemma fails on the Hopf surface (Exercise 6 from the previous series), that is, find a d-exact form which is not $\partial\overline{\partial}$-exact.

Exercise* 5. Let ω be a real $(1,1)$-form such that $\omega(\cdot\,,J\,\cdot)$ is positive definite. Prove that $d\omega = 0$ (i.e. ω is Kähler) if and only if at each point there are holomorphic coordinates $(z_1,\dots,z_n)$ such that writing $\omega = \frac{i}{2}\sum_{i,j} h_{ij} dz_i \wedge \overline{z}_j$ we have $\frac{\partial h_{ij}}{\partial z_k} = \frac{\partial h_{ij}}{\partial \overline{z}_k} = 0$ for all i,j,k.