Complex manifolds, due 20.11.2023
Fubini–Study form

Exercise 1. Put $d^c = \frac{i}{4\pi}(\overline{\partial}-\partial)$ (*Warning:* this definition follows [GH78], [Huy05] takes $d^c = i(\overline{\partial}-\partial)$).
(a) Prove that d^c maps real forms to real forms.
(b) Prove that for a smooth function f we have $d^c f = -\frac{1}{4\pi}\, df \circ J$.
(c) Let (r,θ) be the polar coordinates on $\mathbb{C}^*$, i.e. $z = re^{i\theta}$. Prove that $d^c r = \frac{r}{2}d\theta$, $d^c\theta = -\frac{2}{r}dr$.
(d) Prove that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$.

Exercise 2 (Fubini–Study form)**.** Consider the form $\omega = dd^c \log||z||^2$ on $\mathbb{C}^{n+1}\setminus\{0\}$.
(a) Prove that ω is $\mathbb{C}^*$-invariant, hence induces a 2-form ω_{FS} on $\mathbb{P}^n$, called the *Fubini–Study form.*
(b) Prove that in coordinates $(z_1,\dots,z_n)$ of an affine piece of $\mathbb{P}^n$, we have $\omega_{\mathrm{FS}} = dd^c\log(1+\sum_i |z_i|^2)$.
(c) Prove that ω_{FS} is a Kähler form. Deduce that *all projective manifolds are compact Kähler.*
(d) Let $\varphi \in \mathrm{Aut}\,(\mathbb{P}^n)$ be an automorphism of $\mathbb{P}^n$ given by a unitary matrix. Prove that $\varphi^*\omega_{\mathrm{FS}} = \omega_{\mathrm{FS}}$.
(e) Prove that the restriction of ω_{FS} to a hyperplane $\mathbb{P}^{n-1}\subseteq\mathbb{P}^n$ is again the Fubini–Study form.

Exercise 3. Let ω be a Kähler form on a compact complex manifold X of dimension n, and let $g = \omega(J\,\cdot\,,\,\cdot\,)$ be the associated Riemannian metric.
(a) Prove that for any vector v, the vectors v and Jv are g-orthogonal.
(b) Prove that for any complex submanifold $Y\subseteq X$ of dimension d, the restriction $\frac{1}{d!}\omega^d|_Y$ is the volume form induced by the Riemannian metric $\omega(\,\cdot\,,J\,\cdot\,)$.
(c) Prove that $\mathbb{R}[x]/(x^{n+1}) \ni x \mapsto [\omega] \in H^*(X,\mathbb{R})$ is a degree 2 injective ring homomorphism. In particular, the even Betti numbers of X cannot vanish.

Exercise 4. (a) Prove that the total volume of $\mathbb{P}^1$ with respect to the Fubini–Study form is 1.
(b) Prove that the same holds for $\mathbb{P}^n$ for arbitrary n (*Hint:* use part (a), Exercise 2(e), and induction).
(c) Deduce that the class $[\omega_{\mathrm{FS}}]$ generates $H^2(\mathbb{P}^n,\mathbb{Z})$. In particular, it is the first Chern class of $\mathcal{O}_{\mathbb{P}^n}(1)$, and a Poincaré dual to the hyperplane.
(d) Compute the volume of a smooth hypersurface of degree d.

Exercise 5. Prove that a projective manifold contains a smooth hypersurface which represents a nonzero integral homology class.

Exercise 6 (A non-Kähler complex manifold)**.** Define a $\mathbb{Z}$-action on $\mathbb{C}^2\setminus\{0\}$ by $k\cdot(z,w) = (2^{-k}z, 2^{-k}w)$. The quotient $X = (\mathbb{C}^2\setminus\{0\})/\mathbb{Z}$ is called the *Hopf surface.*
(a) Prove that the above action is proper and free, hence the quotient X is indeed a complex manifold.
(b) Prove that X is diffeomorphic to $\mathbb{S}^1\times\mathbb{S}^3$.
(c) Conclude that X is non-Kähler.

Exercise* 7 (A non-projective compact Kähler manifold)**.** Let $\Lambda\subseteq\mathbb{C}^2$ be a lattice, and let X be a complex torus $\mathbb{C}^2/\Lambda$. Let (z_1,z_2) be the holomorphic coordinates on $\mathbb{C}^2$.
(a) Prove that X is a compact Kähler manifold, with a Kähler form $\frac{i}{2}\sum_i dz_i\wedge d\overline{z}_i$.
(b) Fix a basis $\gamma_1,\dots,\gamma_4$ of $\Lambda = H_1(X;\mathbb{Z})$ and let $a_{ij} = \int_{\gamma_i} dz_j \in \mathbb{C}$. Prove that there is a basis $\gamma_i\wedge\gamma_j$ of $H_2(X,\mathbb{Z})$ such that $\int_{\gamma_i\wedge\gamma_j} dz_1\wedge dz_2 = \det\left[\begin{smallmatrix} a_{i1} & a_{j1}\\ a_{i2} & a_{j2}\end{smallmatrix}\right]$.
(c) Prove that for a generic choice of Λ, the above determinants are $\mathbb{Z}$-linearly independent, so for a cycle $Z\in H_2(X,\mathbb{Z})$ we have $\int_Z dz_1\wedge dz_2 = 0$ if and only if $[Z]=0$.
(d) Using Exercise 5, conclude that for a generic choice of Λ, the torus X is non-projective.

Remark: Here X is deformation-equivalent to a projective manifold. See [Voi04] for a beautiful construction of compact Kähler manifolds that are not even homotopically equivalent to projective ones.

References

[GH78] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978.
[Huy05] D. Huybrechts, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005, An introduction.
[Voi04] C. Voisin, *On the homotopy types of compact Kähler and complex projective manifolds*, Invent. Math. **157** (2004), no. 2, 329–343.