Complex manifolds, due 13.11.2023

$\partial\overline{\partial}$-lemmas

Last time, we did exercises 8–10 from series due 15.10. Next time (13.10), you can still declare exercises from series due 15.10, 6.11, and this one. Since there is a lot of exercises from previous series, this one is shorter.

Recall that a smooth form α is called d- (respectively, ∂-, $\overline{\partial}$-) *closed* if $d\alpha = 0$ (resp. $\partial\alpha = 0$, $\overline{\partial}\alpha = 0$), it is called d- (resp. ∂-, $\overline{\partial}$-) *exact* if there is a form β such that $d\alpha = \beta$ (resp. $\partial\alpha = \beta$, $\overline{\partial}\alpha = \beta$).

Exercise 1. Prove that $\overline{\partial}\overline{\alpha} = \overline{\partial\alpha}$. Infer that a real (p, p)-form is ∂-closed (exact) if and only if it is $\overline{\partial}$-closed (exact).

Exercise 2 (∂-lemma)**.** Prove that for $p \geqslant 1$, any ∂-closed (p, q)-form on a polydisc is ∂-exact

Exercise 3 ($\partial\overline{\partial}$-lemma)**.** Let $p, q \geqslant 1$ and let α be a d-closed (p, q)-form on a polydisc $U \subseteq \mathbb{C}^n$.

(a) Prove that there is a $(p-1, q-1)$-form β on U such that $\alpha = \partial\overline{\partial}\beta$
(*Hint:* combine all the Poincaré lemmas you know: first for d, next for ∂ and $\overline{\partial}$).

(b) Compute β for α being the standard Kähler form $\sum_i dx_i \wedge dy_i$ on $\mathbb{C}^n$.