Complex manifolds, due 06.11.2023
Linear algebra

Last time (23.10), we did Exercises 1–5. Next time, we will continue series due 23.10. I especially encourage you to take a look at Exercises 10–15, concerning applications of the adjunction formula.

Notation. In the following exercises, V is a *real* vector space of dimension $2n$, equipped with an *almost complex structure* $J \in \mathrm{End}_{\mathbb{R}}(V)$, i.e. an endomorphism such that $J^2 = -\mathrm{id}_V$. We write $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and deote the extension of J to $\mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ by the same letter.

Exercise 1. With the above notation, prove the following.

(a) The formula $i \cdot v = Jv$ endows V with a structure of a complex vector space.
(b) The eigenvalues of $J \in \mathrm{End}_C(V_{\mathbb{C}})$ are i and $-i$. Let $V^{1,0}$, $V^{0,1}$ be the corresponding eigenspaces.
(c) The composition of the inclusion $V \hookrightarrow V_{\mathbb{C}}$ with the projection onto $V^{1,0}$ (resp. $V^{0,1}$) is a $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) isomorphism, given by the formula $v \mapsto \frac{1}{2}(v - iJv)$ (resp. $v \mapsto \frac{1}{2}(v + iJv)$).

Exercise 2. Define $\bigwedge^{p,q} V = \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1}$.

(a) Prove that there is a natural isomorphism $\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$.
(b) Prove that the complex conjugation is a ($\mathbb{C}$-antilinear) isomorphism $\bigwedge^{p,q} V \cong \bigwedge^{q,p} V$.

Exercise 3. Fix $x_1, \dots, x_n \in V$. Put $y_j = J(x_j) \in V$, and assume that $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ is an $\mathbb{R}$-basis of V. Define $z_j = \frac{1}{2}(x_j - iy_j) \in V_{\mathbb{C}}$

(a) Prove that $\{z_1, \dots, z_n\}$ and $\{\overline{z}_1, \dots, \overline{z}_j\}$ are $\mathbb{C}$-bases of $V^{1,0}$ and $V^{0,1}$, respectively.
(b) Prove that for every $m \in \{1, \dots, n\}$, we have

$$(2i)^m \cdot (-1)^{\frac{1}{2}m(m+1)} (z_1 \wedge \dots \wedge z_m) \wedge (\overline{z}_1 \wedge \dots \wedge \overline{z}_m) = (x_1 \wedge y_1) \wedge \dots \wedge (x_m \wedge y_m).$$

(c) Let $\{x^1, \dots, x^n, y^1, \dots, y^n\}$ be dual basis of V^*, and let $z^j = x^j + iy^j \in V_{\mathbb{C}}$. Prove that $\{z^1, \dots, z^n\}$ and $\{\overline{z}^1, \dots, \overline{z}^n\}$ are dual bases of $(V^{1,0})^*$ and $(V^{0,1})^*$, respectively.
(d) Let $\omega \coloneqq \sum_{j=1}^n x^j \wedge y^j$ be the standard Kähler form; and let $\Omega \coloneqq z^1 \wedge \dots \wedge z^n$ be the standard holomorphic volume form. Prove that

$$(*) \qquad \tfrac{1}{n!}\omega^n = \left(\tfrac{i}{2}\right)^n \cdot (-1)^{\frac{1}{2}n(n+1)} \Omega \wedge \overline{\Omega}.$$

Exercise 4. Prove that a 2-form $\omega \in \bigwedge^2 V^*$ lies in the space $\omega \in \bigwedge^{1,1} V^*$ if and only if it is J-compatible, i.e. satisfies $\omega(J\,\cdot\,, J\,\cdot\,) = \omega$.

Exercise 5 (Hodge star)**.** Let (W, g) be an oriented euclidean vector space of dimension m. Recall that g extends to an inner product on each $\bigwedge^k W$ in such a way that, if $\{e_1, \dots, e_m\}$ is an orthonormal basis of W, then $\{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant m\}$ is an orthonormal basis of $\bigwedge^k W$.

(a) Put $\mathrm{vol} \coloneqq e_1 \wedge \dots \wedge e_m \in \bigwedge^m W$ for some oriented orthonormal basis $\{e_1, \dots, e_m\}$. Prove that this definition does not depend on the choice of the basis.
(b) Define the *Hodge star operator* $*\colon \bigwedge^k W \longrightarrow \bigwedge^{m-k} W$ by $\alpha \wedge *\beta = g(\alpha, \beta) \cdot \mathrm{vol}$. Prove that on $\bigwedge^k W$ we have $g(\alpha, *\beta) = (-1)^{k(m-k)} g(*\alpha, \beta)$ and $*^2 = (-1)^{k(m-k)} \mathrm{id}_{\bigwedge^k W}$ (*Hint:* it is enough to check that these formulas are satisfied by elements of some orthonormal basis).
(c) Deduce that $*$ is an isometry.