Complex manifolds, due 23.10.2023
Line bundles, Divisors, and Adjunction

Last time (16.10), we did Exercises 1, 2, 5, 7, and started Exercise 3, which we will finish next time (23.10). You can still declare the remaining exercises from the series due 16.10.

Definitions, see [Huy05, §2.3]**.** Let X be a compact complex manifold. A *hypersurface* in X is a closed subset $D \subseteq X$, locally given by a zero of a holomorphic function. A hypersurface is *irreducible* if it cannot be written as a (nontrivial) union of two hypersurfaces. A *divisor* is a finite sum $\sum_i a_i D_i$, where $a_i \in \mathbb{Z}$ and D_i are hypersurfaces in X. A divisor is *effective* if all its coefficients a_i are non-negative. The group of divisors on X is denoted by $\operatorname{Div}(X)$.

Let f be a meromorphic function. The *order* of f along an irreducible hypersurface D is the number $\operatorname{ord}_D(f)$ such that if f is a local equation of D at a smooth point x, then $g = f^{\operatorname{ord}_D(f)} \cdot u$ for some invertible holomorphic function u. One can show that this definition depends only on f and D, cf. [Huy05, Remark 2.3.6(ii)]. The *principal divisor* associated to f is $\operatorname{div}(f) = \sum_D \operatorname{ord}_D(f) D$.

We say that two divisors D, D' are *linearly equivalent*, and write $D \sim D'$, if $D - D'$ is principal. The set of effective divisors linearly equivalent to D is called the *linear system* of D, and denoted by $|D|$.

We denote by $\mathcal{K}^*$ the sheaf of meromorphic functions, and by $\mathcal{O}_X(D)$ the subsheaf of $\mathcal{K}^*$ consisting of those meromorphic functions f such that $D + \operatorname{div}(f)$ is effective.

Exercise 1. Prove that $\mathcal{O}_X(D)$ is in fact (the sheaf of sections of) the line bundle L defined as follows. Let $X = \bigcup_i U_i$ be an open covering such that on U_i, the divisor $U_i \cap D$ is given by equation $f_i \in \mathcal{K}^*(U_i)$ (more precisely: $U_i \cap D = \sum_j a_j D_j$ for hypersurfaces $D_j = \{g_j = 0\}$, and $f_i = \prod_j g_j^{a_j}$). Now, on $U_i \cap U_j$ write $f_i = u_{ij} \cdot f_j$ for some nonvanishing holomorphic function u_{ij}, and let L be the line bundle whose trivializations $\psi_i \colon L|_{U_i} \longrightarrow U_i \times \mathbb{C}$ are related by transition maps $\psi_i \circ \psi_j^{-1} = (\mathrm{id}_{U_i \cap U_j}, u_{ij} \cdot \mathrm{id}_{\mathbb{C}})$.

Exercise 2. (a) Let p be a point in $\mathbb{P}^1$. Prove that $\mathcal{O}_{\mathbb{P}^1}(p) \cong \mathcal{O}_{\mathbb{P}^1}(1)$.

(b) More generally, for points $p_1, \dots, p_k \in \mathbb{P}^1$ prove that $\mathcal{O}_{\mathbb{P}^1}(\sum_{i=1}^k p_k) \cong \mathcal{O}_{\mathbb{P}^1}(k)$.

(c) Let H be a hyperplane in $\mathbb{P}^n$. Prove that $\mathcal{O}_{\mathbb{P}^n}(H) \cong \mathcal{O}_{\mathbb{P}^n}(1)$.

Exercise 3. Prove that $\mathcal{O}_X(0) \cong \mathcal{O}_X$ and $\mathcal{O}_X(-D)$ is the dual of $\mathcal{O}_X(D)$.

Exercise 4. Prove that each element of $|D|$ can be seen as the zero locus of a global section of $\mathcal{O}_X(D)$. Thus $|D|$ can be identified with the projective space $\mathbb{P}(H^0(\mathcal{O}_X(D)))$, in particular $\dim |D| = h^0(D) - 1$.

Exercise 5. Let $\iota \colon D \hookrightarrow X$ be an inclusion of a smooth hypersurface. Construct an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \iota^* \mathcal{O}_D \longrightarrow 0.$$

Exercise 6. Prove that the map $\operatorname{Div}(X)/_\sim \longrightarrow \operatorname{Pic}(X)$ given by $D \mapsto \mathcal{O}_X(D)$ is an injective group homomorphism, whose image contains all line bundles admitting global sections (which are not identically zero). (*Remark:* we will see later that if X is projective, this map is an isomorphism).

Exercise 7. Let $f \colon Y \longrightarrow X$ be a surjective holomorphic map. For a divisor D on X, define its pullback $f^* D$ by pulling back its local equations. Prove that $\mathcal{O}_Y(f^* D) \cong f^* \mathcal{O}_X(D)$.

Exercise 8. For a divisor $D = \sum_{i=1}^k a_i p_i$ on a compact Riemann surface C we put $\deg D = \sum_{i=1}^k a_i$.

(a) Prove that $\deg D$ is invariant under linear equivalence.

(b) Let $\operatorname{Pic}^0(C)$ be the subgroup of $\operatorname{Pic}(C)$ consisting of (line bundles corresponding to) divisors of degree 0. Prove that $\operatorname{Pic}^0(C)$ is trivial if and only if $C \cong \mathbb{P}^1$.

(c) Let $C \subseteq \mathbb{P}^2$ be a smooth cubic, with an inflection point p_0. It is a fact that C admits the following group law: $p_0 = 0$, and for $p, q \in C$, the point $-(p+q)$ is the third common point of C with the line joining p with q. Prove that $C \ni p \mapsto \mathcal{O}_C(p - p_0) \in \operatorname{Pic}^0(C)$ is an isomorphism of groups.

Exercise 9. Let $Y \subseteq X$ be a smooth submanifold of a compact complex manifold X.

(a) Prove that the tangent bundle T_Y is a subbundle of $T_X|_Y$. The quotient $(T_X|_Y)/T_Y$ is called the *normal bundle* of Y, and is denoted by $\mathcal{N}_{Y/X}$.

(b) Let $f \colon X \longrightarrow B$ be a holomorphic submersion, i.e. a holomorphic map whose differential is surjective at every point. Prove that the normal bundle to each fiber of f is trivial.

(c) Give an example of a smooth fiber of a holomorphic map whose normal bundle is non-trivial.

Recall that the *canonical line bundle* of X is $\det(T^*_X)$. We denote the corresponding divisor by K_X. If X is a compact Riemann surface then $\deg K_X = -\chi(X)$.

Exercise 10. Let D be a smooth hypersurface in X.

(a) Prove that $\mathcal{O}_X(D)|_D$ is isomorphic to the normal bundle $\mathcal{N}_{D/X}$.
(b) Deduce the *adjunction formula* $K_D = (K_X + D)|_D$.
(*Hint:* prove that for an exact sequence $0 \to A \to B \to C \to 0$ of vector spaces we have a natural isomorphism $\det(B) \cong \det(A) \otimes \det(C)$. Now apply this to the exact sequence defining $\mathcal{N}_{D/X}$).

Exercise 11. Let $C \subseteq \mathbb{P}^2$ be a smooth curve of degree d.

(a) Prove that the canonical bundle on C is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^2}(d-3)$.
(b) Deduce (once again) that the genus of C equals $\frac{1}{2}(d-1)(d-2)$.

Exercise 12. Let $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth curve, given by a polynomial which is homogeneous of degree a, b in coordinates of each $\mathbb{P}^1$. Prove that C has genus $(a-1)(b-1)$.

Exercise 13. Let $f_1, \dots, f_k \in \mathbb{C}[z_0, \dots, z_n]$ be homogeneous polynomials of degrees $d_1, \dots, d_k$. Assume that $0 \in \mathbb{C}$ is a regular value of the map $(f_1, \dots, f_k)\colon \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}$, so $X \coloneqq \{f_1 = \dots = f_k = 0\}$ is a codimension k submanifold of $\mathbb{P}^n$, called a *complete intersection of degree* $d_1, \dots, d_k$. Using adjunction formula, prove that the canonical bundle of X is the restriction of $\mathcal{O}_{\mathbb{P}^n}(\sum_i d_i - n - 1)$.

Exercise 14 (Twisted cubic)**.** Let C be the image of $\varphi_{\mathcal{O}_{\mathbb{P}^1}(3)}\colon \mathbb{P}^1 \ni [u : v] \mapsto [u^3 : u^2 v : u v^2 : v^3] \in \mathbb{P}^3$.

(a) Prove that C is not a complete intersection in $\mathbb{P}^3$.
(b) Prove that, nonetheless, $C = \{[x : y : z : t] \in \mathbb{P}^3 : xz = y^2,\ z(yt - z^2) = t(xt - yz)\}$.

Exercise 15. Let $\varphi\colon Y \longrightarrow X$ be a blowup at a smooth point, with exceptional divisor $E \cong \mathbb{P}^{n-1}$, where $n = \dim X$. Prove that $\mathcal{O}_Y(E)|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and $K_Y \sim \varphi^* K_X - (n-1)E$.

Let X be a projective manifold. A *curve* in X is an analytic subvariety of dimension 1. For a line bundle L and a smooth curve C, we define the *intersection number* $L \cdot C$ as $\deg L|_C$. There is a unique way to extend this definition to an intersection product between divisors and curves, which agrees with the intersection pairing $H^2(X) \times H_2(X) \longrightarrow \mathbb{Z}$, and satisfies *projection formula* $f^* D \cdot C = D \cdot f_* C$.

If $C \neq D$ are two irreducible curves on a projective surface X, then $C \cdot D$ is the sum of local intersection multiplicities $\dim \mathbb{C}[\![x, y]\!]/(f, g)$, where f, g are equations of C, D in local coordinates (x, y).

Exercise 16. Let C be a curve on a smooth projective surface X. Its *arithmetic genus* is $p_a(C) = \frac{1}{2} C \cdot (K_X + C) + 1$. Prove that if C is smooth then $p_a(C)$ is the usual genus of C.

Exercise 17 (Resolution of plane curve singularities)**.** Let C be a curve on a smooth surface X, let $x \in C$ be a point of multiplicity μ, and let $\varphi\colon Y \longrightarrow X$ be a blowup at x, with exceptional divisor E. Let $\widetilde{C}$ be the proper transform of C, i.e. the closure of the preimage of $C \setminus \{x\}$.

(a) Prove that we have linear equivalences $\varphi^* K_X \sim K_Y - E$ and $\varphi^* C \sim \widetilde{C} + \mu E$.
(b) Prove that $E^2 = -1$, $\widetilde{C} \cdot E = \mu$, $\widetilde{C}^2 = C^2 - \mu^2$, $\widetilde{C} \cdot K_Y = C \cdot K_X + \mu$ and $p_a(\widetilde{C}) = p_a(C) - \frac{1}{2}\mu(\mu - 1)$.
(c) Deduce from Exercise 16 that after finitely many blowups over x, the proper transform of C becomes smooth (in fact, we get this way the normalization of C).
(d) Deduce that $p_a(C)$ is a nonnegative integer.

Exercise 18. Let D be a divisor such that the line bundle $\mathcal{O}_X(D)$ is generated by global sections (we say that $|D|$ is *base point free*). Let $\varphi_{|D|}$ be the corresponding map $X \longrightarrow \mathbb{P}^N$.

(a) Prove that the linear system $|D|$ consists of pullbacks of hyperplane sections of $\varphi_{|D|}(X) \subseteq \mathbb{P}^N$.
(b) Let $C \subseteq X$ be a curve. Prove that $C \cdot D \geqslant 0$, and $C \cdot D = 0$ if and only if $\varphi_{|D|}$ maps C to a point.

Exercise 19 (Continuation of Exercise 5 from the previous series)**.** Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ be a Hirzebruch surface for some $n \geqslant 1$. Let F be a fiber, and let C be the section with $C^2 = -n$.

(a) Prove that every curve $L \not\supseteq C$ satisfies $L^2 \geqslant 0$. Deduce that $\mathbb{F}_n \not\cong \mathbb{F}_m$ for $n \neq m$.
(b) Prove that the line bundle $\mathcal{O}_{\mathbb{F}_n}(1)$ is isomorphic to $\mathcal{O}_{\mathbb{F}_n}(nF + C)$. Deduce (once again) that the associated map to $\mathbb{P}^{n+1}$ contracts only C.
(c) Prove that we have a linear equivalence $K_X \sim -(n+2)F - 2C$.

Exercise 20. Let C be an irreducible curve on a smooth projective surface X such that $K_X \cdot C < 0$ and $C^2 \leqslant 0$. Prove that $C \cong \mathbb{P}^1$ and $C^2 \in \{-1, 0\}$. (*Hint:* use Exercise 17(d) and (b)).