Complex manifolds, due 16.10.2023

Projective space, line bundles and Grassmanians

General rules. Each week, there will be a series of exercises. In the beginning of the classes, please declare which ones you would like to present at the blackboard. Note that there usually will be more exercises than we will be able to discuss: *you are not expected to solve all of them, please choose one or two from each series which suits you best.* Of course, you are also encouraged to present partial solutions, comments, etc. The exercises marked with an asterisk are not necessarily more difficult, but they are less directly related to the lecture, so I suggest focusing more on the ones without asterisk.

Notation. Today, all manifolds and bundles are holomorphic, unless stated otherwise. For a vector bundle E we denote by $H^0(E)$ the vector space of its global sections.

Recall that $\mathcal{O}_{\mathbb{P}^n}(-1)$ denotes the tautological line bundle on $\mathbb{P}^n$. We denote its dual by $\mathcal{O}_{\mathbb{P}^n}(1)$, and write $\mathcal{O}_{\mathbb{P}^n}(k) = \mathcal{O}_{\mathbb{P}^n}(k) = \mathcal{O}(1)^{\otimes k}$. Recall that the homogeneous coordinates $z_0, \dots, z_n$ define global sections of $\mathcal{O}(1)$. In the exercises below, you will need the following fact [Huy05, Proposition 2.4.1]:

Proposition. The sections $z_0, \dots, z_n$ span the vector space $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. As a consequence, the space $H^0(\mathcal{O}_{\mathbb{P}^n}(k))$ can be identified with the space of homogeneous polynomials of degree k in n variables.

Excercise 1. Let L be a line bundle. Show that L is trivial if and only if both L and its dual L^* admit global sections which are not identically zero.

Excercise* 2. Show that the tautological line bundle over $\mathbb{P}^n$ with zero section removed can be identified with $\mathbb{C}^{n+1} \setminus \{0\}$. Use this to construct a $\mathcal{C}^\infty$ fibration $\mathbb{S}^{2n-1} \longrightarrow \mathbb{P}^n$ with fiber $\mathbb{S}^1$: for $n=1$, one recovers this way the classical *Hopf fibration*.

Excercise 3 (Globally generated line bundles give morphisms to $\mathbb{P}^N$)**.** Let L be a line bundle over a complex manifold X. Let V be a vector subspace of $H^0(L)$, and let $\mathrm{Bs}(V)$ denote the common zero locus of all elements of V. Let $s_0, \dots, s_N$ be a basis of V.

(a) Show that the formula
$$\varphi_V \coloneqq [s_0 : \dots : s_N] \colon X \setminus \mathrm{Bs}(V) \longrightarrow \mathbb{P}^N$$
defines a holomorphic map such that $\varphi_V^* \mathcal{O}_{\mathbb{P}^N}(1) = L$. Show that φ_V is independent of the choice of the basis of V, up to an automorphism of the target. For $V = H^0(L)$ we simply write $\varphi_V = \varphi_L$.

(b) Fix a point $x \in \mathbb{P}^n$, and let $V = \{\sigma \in H^0(\mathcal{O}(1)) : \sigma(x) = 0\}$. Describe the map φ_V.

(c) The map $\varphi_{\mathcal{O}_{\mathbb{P}^1}(n)} \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ is called the *n-th Veronese embedding*. Describe it for $n = 1, 2$.

(d) Describe the restriction of the map $\varphi_V \colon \mathbb{P}^2 \longrightarrow \mathbb{P}^1$ from (b) to the image of the second Veronese embedding (there are two cases to consider: either the point $x \in \mathbb{P}^2$ lies on that image or not).

Excercise* 4 (Euler sequence, cf. [Huy05, 2.4.4] or [GH78, p. 409])**.** Prove that on $\mathbb{P}^n$ we have an exact sequence of vector bundles
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \bigoplus_{j=0}^{n} \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$
where $T_{\mathbb{P}^n}$ is the tangent bundle to $\mathbb{P}^n$; the first map is given by global sections $z_0, \dots, z_n$, and j-th coordinate of the second map is given by $1 \mapsto \frac{\partial}{\partial z_j}$. Describe the dual to this sequence, and use it to prove that any holomorphic n-form on $\mathbb{P}^n$ has a pole along a hypersurface of degree $n+1$ (last time we proved it for $n=2$ by an explicit computation).

Excercise 5. Let E be a vector bundle on a complex manifold X. Repeating fiberwise the definition of $\mathbb{P}^n$ and $\mathcal{O}_{\mathbb{P}^n}(-1)$, define the *projective bundle* $\mathbb{P}(E) \longrightarrow X$, whose fiber over $x \in X$ is the projectivization of the fiber E_x; and the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$. Prove that for any line bundle L on X we have an isomorphism $\mathbb{P}(E \otimes L) \cong \mathbb{P}(E)$.

Warning: some authors call $\mathbb{P}(E)$ what we call $\mathbb{P}(E^*)$, see e.g. [Har77, p. 162].

Excercise 6 (Hirzebruch surfaces)**.** Fix an integer $n \geqslant 0$ and let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$.

(a) Prove that $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_1$ is a blowup of $\mathbb{P}^2$ at a point.

(b) Prove that $\mathbb{F}_n$ is isomorphic to the hypersurface $\{x_0^n y_1 = x_1^n y_2\} \subseteq \mathbb{P}^1 \times \mathbb{P}^2$, where $[x_0 : x_1]$ and $[y_0 : y_1 : y_2]$ are homogeneous coordinates on $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively.

(c) Let $C = \mathbb{P}^1 \times \{[1:0:0]\}$ and $S_p = \{([u:v], [p(u,v) : v^n : u^n]) : [u:v] \in \mathbb{P}^1\}$ for homogeneous $p \in \mathbb{C}[u,v]$ of degree n. Prove that C and S_p are sections of the $\mathbb{P}^1$-bundle $\mathbb{F}_n \longrightarrow \mathbb{P}^1$, such that $C \cap S_p = \emptyset$ and, for general p, q, S_p meets S_q normally in n points.

(d) Let $\mathcal{O}_{\mathbb{F}_n}(1)$ be the dual to the tautological bundle defined in Exercise 5, and let $\varphi = \varphi_{\mathcal{O}_{\mathbb{F}_n}(1)}$ be the map from Exercise 3(a). Prove that $\varphi(\mathbb{F}_n)$ is a cone over the image of the n-th Veronese embedding $\varphi_{\mathcal{O}(n)} \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^n$, see Exercise 3(c), with vertex at $\varphi(C)$. Draw pictures for $n = 0, 1, 2$.

The remaining parts require some background in algebraic topology.

(e)* Prove that $H_2(\mathbb{F}_n; \mathbb{Z})$ is freely generated by the classes of C and the fiber F; and the intersection form is given by $F^2 = 0$, $F \cdot C = 1$, $C^2 = -n$ (*Hint:* to get the last equality, prove that the section S_p from (c) satisfies $S_p^2 = n$, $S_p \cdot C = 0$, and write S_p as a linear combination of C and F).

(f)* Up to a diffeomorphism, there are exactly two $\mathbb{S}^2$-bundles over $\mathbb{S}^2$ (indeed, they are classified by $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, cf. discussion of clutching functions in [Hat17, p. 22]). Prove that $\mathbb{F}_n$ is diffeomorphic to the trivial one if n is even, and to the non-trivial one if n is odd.

Excercise 7 (Grassmannian)**.** Let $\mathrm{Gr}_k(V)$ be the set of k-dimensional subspaces of a vector space V.

(a) Say that $V = \mathbb{C}^n$. For a subspace $W \subseteq \mathbb{C}^n$, choose its basis and write its elements in an $n \times k$ matrix of rank k. Let $M_{n,k}$ be the set of such matrices. Show that $\mathrm{Gr}_k(\mathbb{C}^n)$ can be identified with the quotient of $M_{n,k}$ by the left action of $\mathrm{Gl}_n(\mathbb{C})$. This way, $\mathrm{Gr}_k(\mathbb{C}^n)$ becomes a compact topological space.

(b) For a k-element subset $I \subseteq \{1, \dots, n\}$, let M_I be the i-th $k \times k$ minor of a matrix $M \in M_{n,k}$. Prove that $U_I \coloneqq \{[M] : \det(M_I) \neq 0\}$ is a well-defined, open subset of $\mathrm{Gr}_k(\mathbb{C}^n)$; and $\mathrm{Gr}_k(\mathbb{C}^n) = \bigcup_I U_I$ is an open covering.

(c) Show that every class $[M] \in U_I$ admits a unique representative N such that $N_I = \mathrm{id}_{k \times k}$. Let N_I' be the matrix obtained from N by removing the minor N_I. Show that the map $\varphi_I \colon U_I \longrightarrow \mathrm{Mat}((n-k) \times k, \mathbb{C}) \cong \mathbb{C}^{(n-k)k}$ given by $\varphi_I[M] = N_I'$ is a homeomorphism.

(d) Prove that the collection $\{U_I, \varphi_I\}$ defines a holomorphic atlas on $\mathrm{Gr}_k(\mathbb{C}^n)$. This way, $\mathrm{Gr}_k(\mathbb{C}^n)$ becomes a compact complex manifold of dimension $(n-k)k$.

(e) Prove that $\mathrm{Gr}_1(V) \cong \mathbb{P}(V)$ and $\mathrm{Gr}_{\dim V - 1}(V) \cong \mathbb{P}(V^*)$.

Excercise* 8 (Plücker embedding)**.** Let V be a complex vector space. To a k-dimensional subspace $W \subseteq V$, we associate a line in $\bigwedge^k V$ spanned by $w_1 \wedge \dots \wedge w_k$ for any basis $\{w_1, \dots, w_k\}$ of W. Prove that the resulting map $\mathrm{Gr}_k(V) \longrightarrow \mathbb{P}(\bigwedge^k V)$ is a holomorphic embedding: it is called the *Plücker embedding*. Conclude that all Grassmanians are projective.

Excercise* 9 (Tautological bundle on $\mathrm{Gr}_k(V)$)**.** Let $S \longrightarrow \mathrm{Gr}_k(V)$ be the sub-bundle of the trivial bundle $V \times \mathrm{Gr}_k(V) \longrightarrow \mathrm{Gr}_k(V)$ whose fiber over a point $W \in \mathrm{Gr}_k(V)$ is the subspace $W \subseteq V$ itself.

(a) Show that S is indeed a holomorphic subbundle: the frame at a point is given by the columns of the matrix N from Exercise 7(c).

(b) Prove that the map $\varphi_{\bigwedge^k S^*} \colon \mathrm{Gr}_k(V) \longrightarrow \mathbb{P}(\bigwedge^k V)$ defines the Plücker embedding.

(c) Let Q be the quotient of the trivial bundle by S. Show that there is a canonical isomorphism $Q \otimes S^* \cong T_{\mathrm{Gr}_k(V)}$. This generalizes the Euler sequence from Exercise 4.

Excercise* 10. Let E be a vector bundle of rank k on X, and let $V \subseteq H^0(E)$ be an n-dimensional subspace such that for every point $x \in X$, the values $\{\sigma(x) : \sigma \in V\}$ span the fiber E_x. As in Exercise 3, define a map $\varphi_V \colon X \longrightarrow \mathrm{Gr}_{n-k}(V)$ such that $\varphi_V^* S^* = E$.

References

[GH78] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, 52.

[Hat17] A. Hatcher, *Vector bundles & K-theory*, https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf, 2017.

[Huy05] D. Huybrechts, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005, An introduction.