Complex manifolds 02.10.2023

A biased revision of Riemann surfaces

Excercise 1 (cf. [Huy05, §2.4]). Consider the complex projective plane $\mathbb{P}^2$ with homogeneous coordinates $[X:Y:Z]$. Let $P \in \mathbb{C}[X,Y,Z]$ be a homogeneous polynomial.

(a) Show that the zero locus of P is a well defined subset of $\mathbb{P}^2$, call it C.

(b) Show that C is a smooth submanifold of $\mathbb{P}^2$ if and only if the derivatives $\frac{\partial P}{\partial X}, \frac{\partial P}{\partial Y}, \frac{\partial P}{\partial Z}$ do not have common zeros (*Hint:* use Euler identity $X\frac{\partial P}{\partial X} + Y\frac{\partial P}{\partial Y} + Z\frac{\partial P}{\partial Z} = (\deg P)\cdot P$)

(c) Let $(x,y) = (\frac{X}{Z}, \frac{Y}{Z})$ be the affine coordinates of the piece $U = \{Z \neq 0\}$ of $\mathbb{P}^2$. Show that the 2-form $dx \wedge dy$ extends to a meromorphic 2-form on $\mathbb{P}^2$, with pole of order 3 along $\{Z=0\}$.

(d) Write $p(x,y) = P(x,y,1) \in \mathbb{C}[x,y]$. Prove that the 1-forms $(\frac{\partial p}{\partial y})^{-1}\cdot dx$ and $-(\frac{\partial p}{\partial x})^{-1}\cdot dy$ glue to a meromorphic form α on C, such that any 1-form $\widetilde{\alpha}$ on U such that $\widetilde{\alpha}_{U\cap C} = \alpha|_{U\cap C}$ satisfies $\widetilde{\alpha}\wedge dp = dx\wedge dy$. We call α the *residue* of $dx \wedge dy$ along C, and write $\alpha = \frac{dx\wedge dy}{dp}$.

(e) Assume that $d \coloneqq \deg P \geqslant 3$, and let $q \in \mathbb{C}[x,y]$ be a polynomial of degree at most $d-3$. Prove that the 1-forms $q\frac{dx\wedge dy}{dp}$ extend to holomorphic 1-forms on C. We will see later that all holomorphic 1-forms on C are of this type, cf. [GH78, p. 148].

(f) Show that if $d \in \{1,2\}$ then C is isomorphic to $\mathbb{P}^1$, so it does not admit holomorphic 1-forms.

Excercise 2 (cf. [Don11, §II.6.1]). Let X be a compact Riemann surface with a nonvanishing holomorphic 1-form α. Prove that X is isomorphic to a quotient of $\mathbb{C}$ by a lattice of rank 2, as follows.

(a) Endow a universal cover $p\colon \widetilde{X} \longrightarrow X$ with a structure of a Riemann surface.

(b) Prove that a map $F\colon \widetilde{X} \longrightarrow \mathbb{C}$ given by $x \mapsto \int_\gamma \alpha$, where γ is a path joining x with a chosen base point, is holomorphic and satisfies $dF = p^*\alpha$. In particular, F is a local homeomorphism.

(c) Prove that F is a covering map, hence an isomorphism since $\mathbb{C}$ is simply connected (*This is a bit technical. As a warm-up, give an example of a local diffeomorphism which is not a covering*).

(d) Infer from compactness of X that the group $\pi_1(X) \subseteq \mathrm{Aut}(\mathbb{C})$ is a lattice of rank 2.

Deduce that a smooth cubic curve on $\mathbb{P}^2$ is isomorphic to a torus $\mathbb{C}/\Lambda$ for some lattice Λ of rank 2 (*Hint*: use the 1-form $\frac{dx\wedge dy}{dp}$ from Exercise 1(e)).

Exercises 3–5, which follow the discussion in [Don11, §4.2.2–4.2.3], give a hands-on topological construction of a normalization of a Riemann surface.

Excercise 3. Let $f\colon X \longrightarrow Y$ be a proper holomorphic map of connected Riemann surfaces, let $\Delta \subseteq Y$ be the image of the zero locus of df, and let $R = f^{-1}(\Delta)$.

(a) Prove that the restriction $f|_{X\setminus R}\colon X\setminus R \longrightarrow Y \setminus \Delta$ is a proper covering. Let d be its degree.

(b) Define the *monodromy representation* $\pi_1(X\setminus\Delta) \longrightarrow S_d$, where S_d is the symmetric group on d letters, as follows: take a loop in Y with base point y, lift it to a path with endpoints in $f^{-1}(y)$, and consider the induced permutation of $f^{-1}(y)$.

(c) Let $p \in \mathbb{C}[x]$ be a polynomial, let $X = \{(x,y)\in\mathbb{C}^2 : y^2 = p(x)\}$, and let $f\colon X \longrightarrow \mathbb{C}$ be the projection onto the first factor. Describe the monodromy representation of f.

Excercise 4. Let Y be a connected Riemann surface, let Δ be a discrete subset of Y, and let $\rho\colon \pi_1(Y\setminus\Delta) \longrightarrow S_d$ be a transitive representation. Construct a proper map of Riemann surfaces $f\colon X \longrightarrow Y$ whose monodromy representation is ρ, as follows.

(a) Consider a simple case when $(Y,\Delta) = (\mathbb{D}, 0)$, and ρ maps the generator $\gamma \in \pi_1(Y\setminus\Delta)$, to a d-cycle. Then $X = \mathbb{D}$, and $f\colon X\longrightarrow Y$ is given by $z\mapsto z^d$.

(b) Construct a local model as above for an arbitrary permutation $\rho(\gamma) \in S_d$.

(c) Define X by patching together $Y\setminus\Delta$ and the above local models. Check carefully that the resulting topological space is Hausdorff, and endow it with a holomorphic structure.

Excercise 5. Let $C \subseteq \mathbb{P}^2$ be a plane curve, i.e. a zero set of an irreducible polynomial P. Let $\operatorname{Sing} C$ be its singular locus, i.e. the common zero set of the derivatives of P.

(a) Apply Exercises 3 and 4 to construct a map $\widetilde{C} \longrightarrow C$ which is an isomorphism away $\operatorname{Sing} C$. We call $\widetilde{C}$ the *normalization* of C.

(b) Prove that the composition $\widetilde{C} \longrightarrow C \hookrightarrow \mathbb{P}^2$ is holomorphic.

(c) Describe the normalization of cubics $\{x^2z = y^3\}$ and $\{x^2z = y^2(y-z)\}$.

(d) Compare this construction with the one via the integral closure of the coordinate ring.

Below, we denote by χ the topological Euler characteristic. Please choose your favorite definition (via triangulations, singular homology, de Rham cohomology, or Poincaré-Hopf formula). The *genus* of a compact Riemann surface X is $1-\frac{1}{2}\chi(X)$.

Excercise 6. Let N be a complex submanifold of a complex manifold M. Prove that $\chi(M) = \chi(M\setminus N)+\chi(N)$. Show that in the real case, the analogous formula holds mod 2.

Excercise 7 (Hurwitz formula)**.** Let $f\colon X\longrightarrow Y$ be a holomorphic map of compact Riemann surfaces. We say that $x\in X$ is a *ramification point of f with index r_x* if locally around x, f is given by $z\mapsto z^{r_x}$. Let d be the *degree* of f, i.e. the number of points in a general fiber. Prove that

$$\chi(X) = d\cdot \chi(Y) - \sum_{x\in X}(r_x-1)$$

by lifting a triangulation of Y to that of X (it is a fact that Y admits a triangulation).

Excercise 8 (Genus–degree formula)**.** Let $C\subseteq \mathbb{P}^2$ be a smooth curve of degree d. Prove that the genus of C equals $\frac{1}{2}(d-1)(d-2)$ (*Hint:* use Hurwitz formula to the map induced by a projection from a point off C). Deduce that not all Riemann surfaces embed as smooth curves in $\mathbb{P}^2$.

Excercise 9. Prove that $\mathbb{P}^1\times\mathbb{P}^1$ contains Riemann surfaces of arbitrary genus. Does every Riemann surface embed into $\mathbb{P}^1\times\mathbb{P}^1$?

Excercise 10. Let X be a compact Riemann surface with a meromorphic form β. We will see later that the *degree of* β, i.e. the number of zeroes minus the number of poles, equals $-\chi(X)$, cf. [Don11, §7.1.2]. Using this fact, give another proof of the Hurwitz formula (Exercise 7) and the genus-degree formula (Exercise 8).

Excercise 11 (Hyperelliptic curves)**.** Let $p\in\mathbb{C}[x]$ be a polynomial of degree d with no multiple roots, and let $g=\lceil \frac{1}{2}d\rceil -1$. Let $C_0=\{y^2=p(x)\}\subseteq\mathbb{C}^2$, and let C be the normalization of the closure $\overline{C}_0\subseteq\mathbb{P}^2$, see Exercise 5.

(a) Assume that $d=2g+2$ is even. Write $p=c\cdot\prod_{i=1}^{d}(x-a_i)$ for some $a_1,\dots,a_d\in\mathbb{C}$, and let $C_0'=\{(\widetilde{x},\widetilde{y})\in\mathbb{C}^2 : \widetilde{y}^2=c\cdot\prod_i(1-a_i\widetilde{x}))\}$. Show that C is isomorphic to the Riemann surface obtained by gluing C_0 with C_0' by $(\widetilde{x},\widetilde{y})\sim(x^{-1},yx^{-g-1})$.

(b) Prove that C is a curve of genus g, as follows. Like before, let $a_1,\dots,a_{2g+2}$ denote the roots of p, with $a_{2g+2}=\infty$ if d is odd. Cut $\mathbb{P}^1$ along some arcs $\gamma_1,\dots,\gamma_g$ joining a_{2i-1} to a_{2i}, and let V be the resulting compact surface with boundary. Now, glue two copies of V along ∂V, and show that the resulting compact surface is diffeomorphic to C.

A Riemann surface isomorphic to the above curve C for $g\geqslant 2$ is called a *hyperelliptic curve*.

Excercise 12. Prove that $\{(x,y)\in\mathbb{C}^2 : y^2=\sin x\}$ is a Riemann surface which cannot be realized as an interior of a compact manifold with boundary (*Hint:* what would be its genus?).

The following Exercises 13–14, taken from [Ara12, §1.6], give explicit examples of the Jacobian variety of a Riemann surface. The same construction in arbitrary dimension, called the *Albanese* variety, will appear later in the lecture. Let us accept the following fact.

Proposition. Let X be a compact Riemann surface of genus g. Then the space $\Omega^1(C)$ of holomorphic 1-forms has dimension g. Let $\alpha_1,\dots,\alpha_g$ be its basis. Let $\widetilde{X}\longrightarrow C$ be the universal cover, choose a base point $x_0\in\widetilde{X}$ and let $f_j\colon\widetilde{X}\longrightarrow\mathbb{C}$ be a map defined by $x\mapsto\int_\gamma\alpha_j$, for any path γ joining x_0 with x. Then the map $(f_1,\dots,f_g)\colon\widetilde{C}\longrightarrow\mathbb{C}^g$ descends to a map $\alpha\colon C\longrightarrow\mathbb{C}^g/\Lambda$ for some lattice Λ.

The variety $\mathbb{C}^g/\Lambda$ is called the *Jacobian* of C, and denoted by $J(C)$. Every holomorphic map from X to an Abelian variety (i.e. manifold of type $\mathbb{C}^n/\Gamma$ for some lattice Γ) factors through α.

Excercise 13. Let C be a compact Riemann surface of genus 3. Define the "Gauss map" $\alpha'\colon C\longrightarrow\mathbb{P}^2$ by assigning to each point of C the image of the derivative of the map $C\longrightarrow J(C)$.

(a) Assume that C is hyperelliptic, given by $y^2=p(x)$ for some polynomial p of degree 7 or 8, see Exercise 11. Using a basis $\frac{dx}{\sqrt{p(x)}},\frac{x\,dx}{\sqrt{p(x)}},\frac{x^2\,dx}{\sqrt{p(x)}}$ of $\Omega^1(C)$, prove that $\alpha'(x,y)=[1:x:x^2]$, i.e. α' is the 2-1 cover $C\longrightarrow\mathbb{P}^1$.

(b) Assume that $C \subseteq \mathbb{P}^2$ is given by a homogeneous polynomial q of degree 4, cf. Exercise 8. Using the basis of $\Omega^1(C)$ given by 1-forms $\frac{dx\wedge dy}{dq}$, $x\frac{dx\wedge dy}{dq}$, $y\frac{dx\wedge dy}{dq}$ from Exercise 1(e), prove that the Gauss map is the original embedding $C \hookrightarrow \mathbb{P}^2$.

(c) Conclude that the quartic in $\mathbb{P}^2$ is not hyperelliptic (for another proof see [Har77, Exercise 3.2]).

Excercise 14. Let C, E be the (hyper)elliptic curves defined by $y^2 = x^6 + 1$ and $v^2 = u^3 - 1$, so C has genus 2 and E has genus 1. Define holomorphic maps $\pi_1, \pi_2 \colon C \longrightarrow E$ by $\pi_1(x, y) = (x^2, y)$, $\pi_2(x, y) = (x^{-2}, \imath \cdot yx^{-3})$, and describe the factorization of $(\pi_1, \pi_2)\colon C \longrightarrow E \times E$ through $J(C)$.

Excercise 15 (Milnor fibration, cf. [Mil68])**.** Let $f \colon \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a holomorphic function such that $f^{-1}(0) = 0$, and 0 is the unique critical point of f. Let $B \subseteq \mathbb{P}^2$ be a closed ball of small radius $\varepsilon > 0$ around the origin, and let $F = f^{-1}(\delta) \cap B$ for some $\varepsilon \gg \delta > 0$ be the *Milnor fiber* of f.

(a) Consider a vector field v on $f^{-1}(\partial\mathbb{D}_\delta)$ such that $f_* v$ is the unit vector field on $\partial\mathbb{D}_\delta$. Prove that v can be chosen in such a way that its time one flow $\varphi\colon f^{-1}(\delta) \longrightarrow f^{-1}(\delta)$ exists and is the identity outside F. In particular, φ restricts to the *monodromy diffeomorphism* $\varphi\colon F \longrightarrow F$, which does not depend on v up to an isotopy.

(b) Consider $f(x, y) = xy$. Prove that F is diffeomorphic to a cylinder $\mathbb{S}^1 \times [0, 1]$, and φ is isotopic to a Dehn twist.

(c) Can a similar picture be drawn for $f(x, y) = x^k y$ for any $k \geqslant 2$? (Note that now each point $(0, y)$ is a critical point of f).

(d) Describe the Milnor fiber F and the monodromy φ for $f(x, y) = x^2 - y^3$.

References

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