Complex manifolds, due 22.01.2024
Kodaira vanishing and Weak Lefschetz

Last time, we finished Exercises 4, 5 from series about Riemann surfaces (due 18.12) and Exercise 7 from the series about Hodge numbers (due 08.01). Next time, we will discuss remaining exercises from those series and start this one, which will probably be the last.

Exercise 1. Let L be an ample line bundle on a compact Kähler manifold X. Prove that $H^i(X, L^*) = 0$ for all $i < \dim_{\mathbb{C}} X$. Deduce that L^* has no global sections. Does it mean that L has a global section?

Exercise 2. Find a very ample line bundle L on a compact Kähler manifold X such that $H^i(X, L) \neq 0$ for some $i > 0$ (*Hint:* recall the solution to Exercise 5 from series about Riemann surfaces (due 18.12)).

Exercise 3. Describe ample and very ample line bundles on:
(a) The product $\mathbb{P}^n \times \mathbb{P}^m$.
(b) A blowup of $\mathbb{P}^2$ at a point.

Exercise 4. Let X be a complete intersection in a projective space. Prove that for $0 < p+q < \dim_{\mathbb{C}} X$, the Hodge number $h^{p,q}(X)$ equals 1 if $p = q$ and 0 otherwise.

Exercise 5. Which complex tori can be realized as complete intersections in $\mathbb{P}^N$?

Exercise 6. Compute the Hodge numbers of a surface of degree d in $\mathbb{P}^3$.

Exercise 7. A projective manifold X is *Fano* if the anticanonical divisor $-K_X$ is ample.
(a) Check that $\mathbb{P}^n$ and hypersurfaces in $\mathbb{P}^n$ of degree at most n are Fano.
(b) Prove that on a Fano manifold X we have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$; and $\operatorname{Pic}(X) \cong H^2(X, \mathbb{Z})$.

Exercise 8. Let Y be a smooth hypersurface in a projective manifold X. Prove that the inclusion map $H_k(Y) \longrightarrow H_k(X)$ is surjective for $k \leqslant \dim_{\mathbb{C}} Y$ and an isomorphism for $k < \dim_{\mathbb{C}} Y$.

Fact. The statement of Exercise 8 is true also for *homotopy* groups.

Exercise* 9. Let $X \longrightarrow Y$ be a holomorphic (non-branched) covering of degree n between compact complex surfaces. Prove that $K_X^2 = nK_Y^2$, $\chi_{\text{top}}(X) = n\chi_{\text{top}}(Y)$ and $\chi(\mathcal{O}_X) = n\chi(\mathcal{O}_Y)$.

Exercise 10 (Godeaux and Campedelli surfaces)**.** Let X be the quotient of the quintic surface $\{x^5 + y^5 + z^5 + w^5 = 0\} \subseteq \mathbb{P}^3$ by the $\mathbb{Z}_5$-action given by $\zeta \cdot [x : y : z : w] = [x : \zeta y : \zeta^2 z : \zeta^3 w]$.
(a) Prove that X is a compact Kähler manifold such that $\pi_1(X) = \mathbb{Z}_5$.
(b) Prove that X is a surface such that the canonical divisor K_X is ample, but $h^{2,0}(X) = h^{1,0}(X) = 0$. What are the remaining Hodge numbers of X?
(c) Construct a surface X with ample K_X, $h^{2,0} = h^{1,0} = 0$, and $\pi_1 = \mathbb{Z}_2^3$, by taking a quotient of a complete intersection of four quadrics in $\mathbb{P}^6$.
(d) Compute K_X^2 in the above examples.

Exercise 11. Let Y be a smooth hypersurface in a projective manifold X of dimension at least 4. Prove that the restriction map $\operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(Y)$ is an isomorphism.