Hodge numbers and some complex surfaces

Last time, we did Exercises 1,2,3 and 9, from series about Riemann surfaces (due 18.12), and found a counterexample to the "only if" part of Exercise 4(a). The "only if" part of Exercise 4(b) is also false, and I invite you to find a counterexample to this, too :). However, the "if" parts are true, and they can be used to solve Exercises 5 and 8 from that series. We will discuss them next time.

Notation: Terms *curve* and *surface* refer to complex manifolds, of *complex* dimension 1 and 2.

Theorem (Carrell–Lieberman)**.** *Let X be compact Kähler manifold admitting a vector field on X whose zero set V is nonempty. Then $h^{p,q}(X) = 0$ if $|p-q| > \dim_{\mathbb{C}} V$*

Exercise 1. Find a holomorphic vector field on X vanishing exactly at one point, where
- (a) $X = \mathbb{P}^n$
- (b) $X = \mathbb{P}^n \times \mathbb{P}^m$
- (c) $X = \mathrm{Gr}_k(\mathbb{C}^n)$
- (d)* X is a quadric in $\mathbb{P}^n$

Conclude that the non-vanishing Hodge numbers of X are exactly $h^{p,p}(X)$ for all $p \in \{0, \dots, \dim X\}$.

(*Hint for (a):* The flow of the required vector field should be a subgroup of $\mathrm{Gl}_{n+1}(\mathbb{C})$ isomorphic to $(\mathbb{C}, +)$. For $n = 1$ one can take $\{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) : t \in \mathbb{C}\}$, for arbitrary n a similar formula should work).

Exercise 2. Compute the Hodge numbers of the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, see Exercise 6 from set 2. (*Hint:* recall that X is diffeomorphic to an $\mathbb{S}^2$-bundle over $\mathbb{S}^2$, so $b_2(X) = 2$. How can it be distributed among $h^{0,2}$, $h^{1,1}$ and $h^{2,0}$?)

Exercise 3. Let X be a Kähler surface, and let $Y \longrightarrow X$ be a blowup at a point. Prove that $h^{1,1}(Y) = h^{1,1}(X) + 1$ and $h^{p,q}(Y) = h^{p,q}(X)$ for $(p,q) \neq (1,1)$.

The following theorem and the construction in Exercise 4 are not directly related to our subject, but will be useful in Exercises 5, 6 and 7 below.

Theorem (Noether formula)**.** *Let X be a compact complex surface. Then $\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \chi_{\mathrm{top}}(X))$, where χ_{top} denotes the topological Euler characteristic.*

Exercise* 4 (Branched coverings)**.** Let X be a complex manifold and let B be a nonzero effective divisor such that $\mathcal{O}_X(B) = L^{\otimes n}$ for some line bundle $p \colon L \longrightarrow X$ and an integer $n \geqslant 2$.

Define an *n-fold covering of X branched along B* as $X' \coloneqq \{t^n = p^* s\} \subseteq L$, where t is the tautological section of $p^* L$; and s is the section of $\mathcal{O}_X(B)$ vanishing along B. Let $B' = p^{-1}(B) \subseteq X'$.
- (a) Prove that $p|_{X' \setminus B'} \colon X' \setminus B' \longrightarrow X \setminus B$ is a holomorphic n-fold covering.
- (b) Assume that B is smooth. Prove that X' is a complex manifold, B' is its complex submanifold, and $p|_{X'} \colon X' \longrightarrow X$ is a holomorphic map which restricts to a biholomorphism $B' \longrightarrow B$.
- (c) As before, assume that B is smooth. Prove that $K_{X'} \sim p^*(K_X + (n-1)L)$.
- (d) Describe a double cover of $\mathbb{P}^1$ branched along $2g+2$ distinct points.
- (e) Describe a double cover of $\mathbb{P}^2$ branched along a conic.

Exercise 5. A compact Kähler surface X is *K3* if $H^1(X, \mathcal{O}_X) = 0$ and $K_X \sim 0$.
- (a) Using Noether formula, compute the Hodge numbers of a K3 surface.
- (b) Prove that the Picard group of a K3 surface is a free abelian group of rank at most 20.
- (c) Prove that a quartic surface in $\mathbb{P}^3$ is K3.
- (d) Prove that a double cover of $\mathbb{P}^2$, branched along a smooth sextic curve, is K3.
- (e)* (Kummer surface) Let $A = \mathbb{C}^2/\Lambda$ be a complex torus of dimension 2, and let $\iota \colon A \longrightarrow A$ be the involution given by $\iota(x) = -x$. Prove that ι has 16 fixed points, whose images in A/ι are ordinary double points, i.e. they are locally given by $\{x^2 = yz\}$. Let $X \longrightarrow A/\iota$ be a blowup at each of those 16 points. Prove that X is a K3 surface.
- (f)* Let $L \subseteq \mathbb{P}^2$ be a union of six general lines, and let $Y \longrightarrow \mathbb{P}^2$ be a double cover branched along L. Let $X \longrightarrow Y$ be a blowup at each singular point of Y. Prove that X is a K3 surface. What happens if the lines are not general, e.g. if three of them meet at the same point?
- (g)* Find some lower bounds on the rank of $\mathrm{Pic}(X)$ in the above examples.

Exercise 6. A compact Kähler surface X is *Enriques* if $H^1(X, \mathcal{O}_X) = 0$ and $2K_X \sim 0$, but $K_X \not\sim 0$.

(a) Compute the Hodge numbers of an Enriques surface.

(b)* Let ι be an involution of $\mathbb{P}^1 \times \mathbb{P}^1$ given by $\iota([x : y], [z : w]) = ([y : x], [w : z])$. Let $B \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth, ι-invariant curve of bi-degree $(4, 4)$, not passing through any fixed point of ι. Let $Y \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover branched along B, let ι_Y be an involution of Y lifting ι; let $Y \longrightarrow \overline{X}$ be a quotient by ι_Y; and let $X \longrightarrow \overline{X}$ be a blowup at each singular point of $\overline{X}$. Prove that Y is a K3, and X is an Enriques surface.

Exercise 7. Compute the Hodge numbers of the Hopf surface (Exercise 6 from set 6)

Exercise 8 (Iwasawa threefold)**.** Let $G \subseteq \mathrm{Gl}_3(\mathbb{C})$ be the group of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $H \subseteq G$ be its subgroup consisting of matrices whose entries belong to the ring $\mathbb{Z}[i]$ of Gaussian integers; and consider the action of H on G by left multiplication.

(a) Prove that the quotient $X = G/H$ is a compact complex manifold of dimension 3.

(b) Find a non-closed holomorphic 1-form on X; and deduce that X admits no Kähler form.

(c) Compute the Betti numbers of X and note that they do not obstruct the existence of Kähler form.

(d) Compute the Hodge numbers of X.