Smearing lemma.

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Lemma 1. *Let* $\varphi\colon M \to N$ *be a morphism of* A*-modules with* N *finitely generated and let* $\mathfrak{p} \in \operatorname{Spec}(A)$ *be a point such that* $\varphi \otimes \mathrm{id}\colon M \otimes_A \kappa(\mathfrak{p}) \to N \otimes_A \kappa(\mathfrak{p})$ *is surjective. Then there exists an* $f \in A \setminus \mathfrak{p}$ *such that* $\varphi\colon M_f \to N_f$ *is surjective as well.*

Proof. Consider first the map $\varphi_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$-modules. By assumption, we have $N_{\mathfrak{p}} = \varphi_{\mathfrak{p}}(M_{\mathfrak{p}}) + \mathfrak{p}N_{\mathfrak{p}}$, so by Nakayama lemma, the map $\varphi_{\mathfrak{p}}$ is surjective.

Let C be the cokernel of φ, so we have an exact sequence $M \to N \to C \to 0$. Localizing at $\mathfrak{p}$ gives us $M_{\mathfrak{p}} \to N_{\mathfrak{p}} \to C_{\mathfrak{p}} \to 0$, but $M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is surjective, so $C_{\mathfrak{p}} = 0$. Now C is an image of a finitely generated A-module N, hence is finitely generated itself. Let $c_1, \ldots, c_r \in C$ be its generators. Then $C_{\mathfrak{p}} = 0$ implies that there exist $f_1, \ldots, f_r \in A \setminus \mathfrak{p}$ such that $f_i c_i = 0$ for $i = 1, \ldots, r$. The element $f = \prod_i f_i$ annihilates all c_i, so $C_f = 0$. Inverting f in the exact sequence above, we get $M_f \to N_f$ surjective. $\square$