Local criterion for flatness.

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Definition 1. *Let A be a ring with an ideal $\mathfrak{a}$ and M be an A-module. We say that M is* $\mathfrak{a}$-adically separated *if the intersection $\bigcap_{k\geq 0} \mathfrak{a}^k M$ is equal to zero. We say that M is* $\mathfrak{a}$-adically ideal-separated *if for every ideal I of A the module $I \otimes_A M$ is $\mathfrak{a}$-adically separated.*

Theorem 2. *Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of noetherian local rings and let M be a finitely generated B-module. Let $\mathfrak{a} \subset \mathfrak{m}$ be an ideal in A. Then M is $\mathfrak{a}$-adically ideal-separated.*

Proof. Choose an ideal $I \subset A$, then I is finitely generated, so there is a surjection $A^{\oplus r} \twoheadrightarrow I$ of A-modules. Tensoring with M over A we get a surjection $M^{\oplus r} \twoheadrightarrow I \otimes_A M$ of B-modules. Therefore the B-module $I \otimes_A M$ is finitely generated, hence is $\mathfrak{a}$-adically separated by Krull's Intersection theorem, e.g. [10.17, Atiyah-Macdonald]. □

Lemma 3. *Let B be ring and $\mathfrak{b}$ be an ideal of B such that $\mathfrak{b}^2 = 0$. Let N be a B-module, such that $N/\mathfrak{b}$ is flat over $B/\mathfrak{b}$ and the multiplication map $\mathfrak{b} \otimes_B N \to N$ is injective. Then N is flat over B.*

Proof. Let $I \subseteq B$ be an ideal and consider the following diagram

$$\begin{array}{ccccccccc} & & (I \cap \mathfrak{b}) \otimes_B N & \longrightarrow & I \otimes_B N & \longrightarrow & I/(I \cap \mathfrak{b}) \otimes_B N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle m_2} & & \downarrow{\scriptstyle m_3} & & \\ 0 & \longrightarrow & (I \cap \mathfrak{b})N & \longrightarrow & IN & \longrightarrow & IN/(I \cap \mathfrak{b})N & \longrightarrow & 0 \end{array}$$

The top row comes from tensoring $0 \to I \cap \mathfrak{b} \to I \to I/(I \cap \mathfrak{b}) \to 0$ with N and the vertical arrows denote multiplication. Since $I/(I \cap \mathfrak{b})$ is naturally an ideal of $B/\mathfrak{b}$ and $N/\mathfrak{b}$ is flat over $B/\mathfrak{b}$, the homomorphism m_3 is injective.

Since $\mathfrak{b}^2 = 0$, the B-modules $I \cap \mathfrak{b}$ and $\mathfrak{b}$ are in fact $B/\mathfrak{b}$-modules. Since $N/\mathfrak{b}$ is flat over $B/\mathfrak{b}$, the morphism $i : (I \cap \mathfrak{b}) \otimes_B N \to \mathfrak{b} \otimes_B N$ is injective. Also $m' : \mathfrak{b} \otimes_B N \to N$ is injective by assumption. Therefore, $m_1 = m' \circ i$ is injective and finally m_2 is injective by the Snake Lemma. Thus $I \otimes_B N \to N$ is an injection for every ideal I, so that N is flat over B. □

Theorem 4 (Local criterion for flatness)**.** *Let A be a ring with an ideal $\mathfrak{a}$ and M be an A-module. Consider the conditions*

1. *M is flat over A.*
2. *$M/\mathfrak{a}$ is flat over $A/\mathfrak{a}$ and the natural multiplication homomorphism $\mathfrak{a} \otimes_A M \to M$ is injective.*
3. *$M/\mathfrak{a}$ is flat over $A/\mathfrak{a}$ and $\mathrm{Tor}_1^A(M, A/\mathfrak{a}) = 0$.*
4. *$M/\mathfrak{a}$ is flat over $A/\mathfrak{a}$ and $\mathrm{Tor}_1^A(M, N) = 0$ for every $A/\mathfrak{a}$-module N.*
5. *for every $k \geq 1$ the module $M/\mathfrak{a}^k$ is flat over $A/\mathfrak{a}^k$.*

Then $1 \implies 2 \implies 3 \implies 4 \implies 5$. *Moreover, if* M *is* $\mathfrak{a}$*-adically ideal separated, then all these conditions are equivalent.*

Proof. $1 \implies 2$. This implication follows from the fact that flatness is preserved by base change.

$2 \implies 3$. From the short exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ we get an exact sequence $0 = \mathrm{Tor}_1^A(A, M) \to \mathrm{Tor}_1^A(A/\mathfrak{a}, M) \to \mathfrak{a} \otimes_A M \to M \to M/\mathfrak{a} \to 0$. By assumption, the homomorphism $\mathfrak{a} \otimes_A M \to M$ is injective, so that $\mathrm{Tor}_1^A(A/\mathfrak{a}, M) = 0$.

$3 \implies 4$. Let N be an $A/\mathfrak{a}$-module and F be a free $A/\mathfrak{a}$-module with an epimorphism onto N, so that we get a sequence $0 \to K \to F \to N \to 0$. Then we obtain a sequence

$$\mathrm{Tor}_1^A(F, M) \to \mathrm{Tor}_1^A(N, M) \to K \otimes_A M \to F \otimes_A M$$

By assumption $\mathrm{Tor}_1^A(F, M) = \mathrm{Tor}_1^A(\bigoplus A/\mathfrak{a}, M) = \bigoplus \mathrm{Tor}_1^A(A/\mathfrak{a}, M) = 0$. Since $M/\mathfrak{a}$ is a flat $A/\mathfrak{a}$-module and K, F are $A/\mathfrak{a}$-modules, the map $K \otimes_A M \to F \otimes_A M$ is injective. Thus $\mathrm{Tor}_1^A(N, M) = 0$.

$4 \implies 5$. For any $k \geq 0$ let $A_k := A/\mathfrak{a}^k$ and $M_k := M/\mathfrak{a}^k$. From the short exact sequence $0 \to \mathfrak{a}^{k-1}/\mathfrak{a}^k \to A_k \to A_{k-1} \to 0$ we get an exact sequence

$$\mathrm{Tor}_1^A(\mathfrak{a}^{k-1}/\mathfrak{a}^k, M) \to \mathrm{Tor}_1^A(A_k, M) \to^{d_k} \mathrm{Tor}_1^A(A_{k-1}, M).$$

Now $\mathrm{Tor}_1^A(\mathfrak{a}^k/\mathfrak{a}^{k+1}, M) = 0$ by assumption, so that d_k is is injective for every $k \geq 1$. Since $\mathrm{Tor}_1^A(A_1, M) = 0$, we get that for every $k \geq 1$ the module $\mathrm{Tor}_1^A(A_k, M)$ is zero.

Now we proceed the proof by induction on k. The base case $k = 1$ follows from assumptions. Suppose that M_k is flat over A_k. The homomorphism

$$m : \mathfrak{a}^k/\mathfrak{a}^{k+1} \otimes_A M \to M/\mathfrak{a}^{k+1}M$$

has kernel $\mathrm{Tor}_1^A(A_k, M) = 0$, thus is injective. Therefore, M_{k+1} is flat over A_{k+1} by Lemma 3 applied to $B := A_{k+1}$, $\mathfrak{b} := \mathfrak{a}^k/\mathfrak{a}^{k+1}$ and $N := M_{k+1}$.

$5 \implies 1$. Let $I \subseteq A$ be an ideal. Since M is $\mathfrak{a}$-adically separated, the morphism $p : I \otimes_A M \to \prod_k I \otimes_A M_k$ is injective. Since M_k is flat over A_k, the homomorphism $I \otimes M_k \to M_k$ is injective for every k, thus $m : \prod_k I \otimes_A M_k \to \prod_k M_k$ is also injective, so that $m \circ p : I \otimes_A M \to \prod_k M_k$ is injective. This homomorphism factors through $I \otimes_A M \to M$, so $I \otimes_A M \to M$ is injective. Since I is arbitrary, M is flat over A. □

Corollary 5. *Let* $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ *be a local homomorphism of noetherian local rings and let* M *be a finitely generated* B*-module. Let* $\mathfrak{a} \subset \mathfrak{m}$ *be an ideal in* A*. Then the conditions of Theorem 4 are equivalent.*

Proof. Directly from Theorem 2. □