Varieties of Commuting Matrices are Unexplored

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For simplicity, everything over $\mathbb{C}$. $\mathbb{M}_d$ denotes $d \times d$ matrices.

Definition

The variety* $C_n(\mathbb{M}_d)$ of n-tuples of commuting $d \times d$ is defined as

$$C_n(\mathbb{M}_d) = \{(x_1, \ldots, x_n) \in (\mathbb{M}_d)^n \mid \forall i, j : x_i x_j = x_j x_i\}.$$

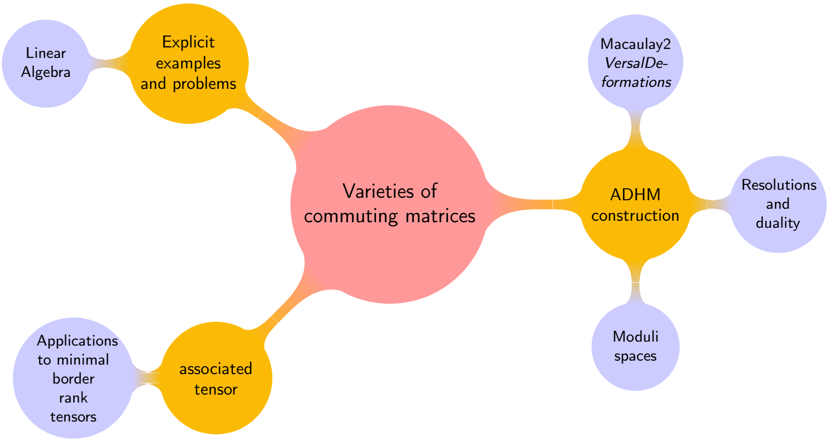
Motzkin, Taussky'55: $n = 2$
... & Stark 2021 $d = 2$
The good open (not necessarily dense!) locus

$$\{(x_1, \ldots, x_n) \in \mathbb{M}_d^n \mid \text{ simultaneously diagonalizable}\}$$

It closure is an irreducible component, *the principal component*.

Subjective big picture

$$C_n(\mathbb{M}_d) = \{(x_1, \ldots, x_n) \in (\mathbb{M}_d)^n \mid \forall i, j : x_i x_j = x_j x_i\}\,.$$



Key open questions, small number of matrices

Classify points of $C_n(\mathbb{M}_d)$ up to $\mathrm{GL}_n \times \mathrm{GL}_d$-action for small n, d.

Find equations for the principal component inside $C_n(\mathbb{M}_d)$.

Is the scheme $C_2(\mathbb{M}_d)$ reduced? Is it Cohen-Macaulay?
(Charbonnel arXiv:2006.12942)

Key open questions II, small number of matrices

What is the smallest d such that $C_3(\mathbb{M}_d)$ is reducible?

Known: $12 \leq d \leq 29$, lower bound Šivic, upper bound Holbrook, Omladič$+\varepsilon$Šivic.

For any d describe (a general point of) any component of $C_3(\mathbb{M}_d)$ other than the principal one. Describe any explicit triple (x_1, x_2, x_3) outside the principal component.

For $(x_1, x_2, x_3) \in C_3(\mathbb{M}_d)$ is it true that $\dim_{\mathbb{C}}(\mathbb{C}[x_1, x_2, x_3] \subset \mathbb{M}_d) \leq d$? (Gerstenhaben's question)

True for the principal component.

Larger number of matrices: nothing (was) known

Classical swindle (e.g. Guralnick):

$$x_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The algebra $\mathbb{C}[x_1, x_2, x_3, x_4]$ is

$$\lambda \cdot Id_4 + \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \lambda \in \mathbb{C}.$$

Violates Gerstenhaber's bound. $C_n(\mathbb{M}_d)$ is reducible for $n, d \geq 4$.

Our results I

Theorem (J-Š)

The number of irreducible components of $C_n(\mathbb{M}_d)$ for $d \leq 7$ is as shown in Table 1; we also have explicit descriptions of general points of each component (and general points are smooth).

	$d \leq 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d \gg 0$
$n \leq 2$	1	1	1	1	1	1	1
$n = 3$	1	1	1	1	1	1	$\gg 0$
$n = 4$	1	1	2	2	2	2	$\gg 0$
$n = 5$	1	1	2	4	4	8	$\gg 0$
$n = 6$	1	1	2	4	7	11	$\gg 0$
$n \geq 7$	1	1	2	4	7	13	$\gg 0$

Table: *Number of components of $C_n(\mathbb{M}_d)$*

Our results I

Typically (not always), the elementary components have the form, up to GL_d action and adding scalar matrices,

$$x_1, \ldots, x_n \in \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$$

where $*$ is $m \times (d-m)$ matrix for some fixed m and for n large enough.

Our results II

Theorem (J-Š)

The variety $C_n(\mathbb{M}_d)$ has generically nonreduced components for all $n \geq 4$ and $d \geq 8$. For example, the locus of quadruples of the form (up to GL_8 action and adding scalar matrices):

$$\begin{bmatrix} 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a generically nonreduced component.

Our results II

Locus $\mathcal{L}$ of quadruples

$$\begin{bmatrix} 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Computer: $\mathcal{L} \cap$ (principal component) $\subset \mathcal{L}$ is a divisor. Get a divisor in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ invariant under S_3 and $GL_4 \times GL_4 \times GL_4$. Which divisor is it? Is it $S_{3333}(\mathbb{C}^4)^{\otimes 3} \subset S^{12}(\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4)$ of Landsberg-Manivel?

Our results III

A *Macaulay2* package *MatricesAndQuot*, which implements most of the basic routines one would like to see. Check out at `https://arxiv.org/src/2106.13137v2/anc/MatricesAndQuot.m2`

1. ADHM construction (matrices to modules and back),
2. torus actions and their limits,
3. primary obstructions.

ADHM construction

Definition

Let $S = \mathbb{C}[y_1, \ldots, y_n]$. For $(x_1, \ldots, x_n) \in C_n(\mathbb{M}_d)$ we define an S-module structure on $\mathbb{C}^d$ by

$$y_i \cdot v := x_i(v) \qquad \text{for all } v \in \mathbb{C}^d.$$

We will denote the resulting *module associated to* $(x_1, \ldots, x_n)$ by M.

The module has tons of invariants: number of generators, Hilbert function, resolution etc. which we employ to get a better grasp on the matrices themselves.

ADHM construction – example

$$x_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad x_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the associated module $M = \langle e_1, e_2, e_3, e_4 \rangle$ we have $x_1(e_3) = e_1$ and so on. The module M is graded, generated by e_3, e_4, with Hilbert series $2 + 2T$ and resolution:

$$\begin{bmatrix} 2 & 6 & 4 & - & - \\ - & - & 4 & 6 & 2 \end{bmatrix}$$

Anyone knows the name/a ref for such "numerically self-dual" modules?

ADHM construction – abstractly

space	objects
$\mathrm{Mod}^d(\mathbb{A}^n)$	modules
$\mathrm{Quot}_r^d(\mathbb{A}^n)$	modules with fixed r generators
$C_n(\mathbb{M}_d)$	modules with fixed basis
$\mathcal{U}^{\mathrm{st}}$	modules with fixed basis and fixed r generators

$$\begin{array}{ccc}
\mathcal{U}^{\mathrm{st}} & \xrightarrow{\text{smooth fib.dim. } rd} & C_n(\mathbb{M}_d) \\
\big\downarrow /\,\mathrm{GL}_d & & \big\downarrow /\,\mathrm{GL}_d \\
\mathrm{Quot}_r^d(\mathbb{A}^n) & \longrightarrow & \mathrm{Mod}^d(\mathbb{A}^n)
\end{array}$$

Table: Moduli spaces

ADHM construction and Quot

Theorem (J-Š)

	$d \leq 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d \gg 0$
$n \leq 2$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$
$n = 3$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$1, \ldots$	$\gg 0$
$n = 4$	$1, \ldots$	$1, \ldots$	$1, 2, \ldots$	$1, 2, \ldots$	$1, 2, \ldots$	$1, 2, \ldots$	$\gg 0$
$n = 5$	$1, \ldots$	$1, \ldots$	$1, 2, \ldots$	$1, 3, 4, \ldots$	$1, 3, 4, \ldots$	$1, 4, 7, 8, \ldots$	$\gg 0$
$n = 6$	$1, \ldots$	$1, \ldots$	$1, 2, \ldots$	$1, 3, 4, \ldots$	$1, 4, 6, 7, \ldots$	$1, 5, 9, 11, \ldots$	$\gg 0$
$n \geq 7$	$1, \ldots$	$1, \ldots$	$1, 2, \ldots$	$1, 3, 4, \ldots$	$1, 4, 6, 7, \ldots$	$1, 6, 10, 12, 13, \ldots$	$\gg 0$

Table: *Number of components of* $\mathrm{Quot}_r^d(\mathbb{A}^n)$*. In each entry, consecutive numbers correspond to the number of components for* $r = 1, 2, \ldots$ *and "*$\ldots$*" means that the numbers stabilize at the value of the last entry. In particular, we see that for* $r \geq 5$ *we already have all the components (for* $d \leq 7$*).*

Connection to tensors

Tensors $t \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. A tensor has rank at most d if it has the form $t = \sum_{i=1}^{d} a_i \otimes b_i \otimes c_i$. A tensor has *border rank* at most d if it is a limit of such. A tensor is *concise* if it does not live in any $V_1 \otimes V_2 \otimes V_3$ where V_i are subspaces, at least one of them proper. Concise tensors have border rank at least d.

Problem (Nightmare)

Classify concise tensors of border rank d.

Problem (Open problem, reasonable)

Do the same for small d. ($d \leq 4$ done although not written down, but already $d = 5$ seems open)

Connection to tensors cd.

For a tuple of matrices $(x_1, \dots, x_{d-1}) \in C_{d-1}(\mathbb{M}_d)$ we may form a *naive* tensor $\sum_{i=1}^{d} e_i \otimes x_i$, where $x_d = Id_d$.

Proposition (Landsberg-Michałek, *Abelian tensors*)

1. *The naive tensor is of minimal border rank iff $(x_1, \dots, x_{d-1})$ is in the principal component,*
2. *Any tensor of minimal border rank which is 1_A-generic is isomorphic to a naive tensor for some $(x_1, \dots, x_{d-1}) \in C_{d-1}(\mathbb{M}_d)$.*

Conclusion: 1_A-generic tensors of minimal border rank correspond to $(x_1, \dots, x_{d-1}) \in C_{d-1}(\mathbb{M}_d)$ in the principal component.

Theorem (J)

For $d \leq 6$, a tuple $(x_1, \dots, x_{d-1}) \in C_{d-1}(\mathbb{M}_d)$ is in the principal component iff $\dim(\mathbb{C}[x_1, \dots, x_{d-1}]) \leq d$.

Macaulay's inverse systems / apolarity for modules.
$S = \mathbb{C}[y_1, \ldots, y_n] \quad T = \mathbb{C}[z_1, \ldots, z_n]$
$F = Se_1 \oplus Se_2 \oplus \ldots \oplus Se_r$
$F^* := Te_1^* \oplus \ldots \oplus Te_r^*$
Is an S-module via $y_i y_j \circ (z_i^2 z_j)e_k^* = z_i e_k^*$. Admits a pairing $F \times F^* \to \mathbb{C}$ defined usually on dual bases.

Theorem (J-Š)

For every $M = F/K$ annihilated by $S_{\gg 0}$ there exist $\sigma_1, \ldots, \sigma_r \in F^*$ such that $K = (S\sigma_1 + \ldots + S\sigma_r)^{\perp}$. Say: M apolar to $\sigma_1, \ldots, \sigma_r$.

Example

The module coming from $x_1, \ldots, x_4$ is the apolar module of $z_1 e_3^* + z_2 e_4^*, z_3 e_3^* + z_4 e_4^*$.

Methods II: Białynicki-Birula decomposition

$\mathbb{G}_{\mathrm{m}} \circlearrowleft \mathrm{Quot}_r^d(\mathbb{A}^n)$.

geometry	algebra
$[F/K]$ is $\mathbb{G}_{\mathrm{m}}$-fixed	$K \subset F$ homogeneous
$\mathrm{Hom}(K, F/K)_i$	$\varphi\colon K \to F/K$ shifting degree by i

Proposition

If $\mathrm{Hom}(K, F/K)_{>0} = 0$ *or* $\mathrm{Hom}(K, F/K)_{<0} = 0$ *locally there exists a retraction onto fixed points.*

Methods II: Białynicki-Birula decomposition – scribbling