Białynicki-Birula decompositions and applications

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What is the Białynicki-Birula decomposition?

For a number $x \in \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ we have

$$\lim_{t\to 0} t \cdot x = \begin{cases} 0 & \text{if } x \neq \infty \\ \infty & \text{if } x = \infty \end{cases}$$

Grouping together the numbers with the same limit, we obtain a decomposition of $\mathbb{RP}^1$ into affine spaces $\mathbb{R}$ and $\{\infty\}$.

The classical *Białynicki-Birula decomposition* [BB73] is a far reaching generalization of this idea. Consider a smooth proper variety $X/\Bbbk$ with an action of $\Bbbk^*$. Let $F_1, \ldots, F_r$ be the components of $X^{\Bbbk^*}$. Let $X_i := \{x \in X \mid \lim_{t\to 0} t \cdot x \in F_i\}$; they are locally closed in X.

> For each i, the limit map $X_i \to F_i$ is regular and in fact it is an affine fiber bundle (=locally on F_i looks like a trivial vector bundle).

The *Białynicki-Birula decomposition* of X is $X^+ = \sqcup_i X_i$.

In the example above, we have $F_0 = X_0 = \{\infty\}$ and $F_1 = \{0\}$, $X_1 = \mathbb{P}^1 \setminus \{\infty\}$. More generally, when X has finitely many $\Bbbk^*$-fixed points, then it decomposes into a union of affine spaces. In particular, every such X is rational.

Functorial interpretation [Dri13]

While incredibly useful, the decomposition above is restricted by the smoothness assumption on X. Drinfeld [Dri13] observed that "the limit map being regular" can be taken as its defining property:

Białynicki-Birula decomposition parameterizes points of X together with their limits at $t = 0$.

This can rephrased rigorously, functorially but quite opaquely as follows. Let $\mathbb{A}^1$ the affine line with its usual $\Bbbk^*$-action. A point of x together with its limit is just an equivariant map $\mathbb{A}^1 \to X$. Therefore, Drinfeld defines X^+ as a functor

$$X^+(S) = \{\varphi\colon \mathbb{A}^1 \times S \to X \mid \varphi \text{ is } \Bbbk^*\text{-equivariant}\}.$$

This comes with the limit map $\pi_X\colon X^+ \to X^{\Bbbk^*}$ and a forgetful map $i_X\colon X^+ \to X$, which are restrictions of φ to $0 \times S$ and $1 \times S$, respectively. The main points here are:

- the map π_X is affine, but not necessarily a bundle,
- the construction is functorial and X^+ is represented by a scheme for all X locally of finite type, see also [AHR15],
- For smooth and proper X we recover the classical BB decomposition above.

Application [Je18]

In the setting very different from smooth varieties, Jelisiejew [Je18] applies the Białynicki-Birula decomposition to the Hilbert scheme of points (which, loosely speaking, parameterizes finite algebras) to *prove* that singularities exist:

> The scheme $\mathrm{Hilb}_{\mathrm{pt}}(\mathbb{A}^{16})$ has arbitrary singularities up to retraction. In particular, it is non-reduced and for all primes p there exist finite algebras over $\mathbb{F}_p$ nonliftable to characteristic zero.

This solves several classical open problems. The main idea is to use the BB decomposition to reduce considerations from the Hilbert scheme to its $\Bbbk^*$-fixed locus.

Decomposition for reductive groups [JS19]

Jelisiejew and Sienkiewicz [JS19] generalized the BB decomposition to groups other than $\Bbbk^*$. Fix a connected linearly reductive affine group $\mathbf{G}$ and an affine monoid $\overline{\mathbf{G}}$ with zero that has $\mathbf{G}$ as group of units. For a $\mathbf{G}$-scheme X define X^+ by

$$X^+(S) = \{\varphi\colon \overline{\mathbf{G}} \times S \to X \mid \varphi \text{ is } \mathbf{G}\text{-equivariant}\}.$$

For the pair $(\mathbf{G}, \overline{\mathbf{G}}) = (\Bbbk^*, \mathbb{A}^1)$ we recover Drinfeld's construction above. For all $\overline{\mathbf{G}}$ and for smooth X, the classical result of Białynicki-Birula generalizes verbatim:

> The variety X^+ is smooth and the morphism
> $$\pi_X\colon X^+ \to X^{\mathbf{G}}$$
> is an affine space fiber bundle with a $\overline{\mathbf{G}}$-action fiber-wise. Moreover, each component of X^+ is a locally closed subvariety of X via i_X.

We can easily check whether a given point lies in a dominant cell:

> If $x \in X^{\mathbf{G}}$ is such that the $\mathbf{G}$-action on $T_{X,x}$ extends to a $\overline{\mathbf{G}}$-action, then $i_X\colon X^+ \to X$ is an open immersion near $x \in X^+$.

Note that for particular choices of $\overline{\mathbf{G}}$, there might be no dominant cell: the orbit of a general point might not compactify to $\overline{\mathbf{G}}$.

Decomposition for the additive group [JS20]

Assume $\mathrm{char}\,\Bbbk = 0$. The unique connected one-dimensional group other that $\Bbbk^*$ is $\mathbb{G}_a := (\Bbbk, +)$. It has a unique smooth equivariant compactification $\mathbb{P}^1$, so for a $\mathbb{G}_a$-scheme X we define the additive Białynicki-Birula decomposition by

$$X^+(S) = \{\varphi\colon \mathbb{P}^1 \times S \to X \mid \varphi \text{ is } \mathbb{G}_a\text{-equivariant}\}.$$

We have analogues of π_X and i_X, where π_X takes the limit at ∞. Assume X is projective. Then the following properties hold:

1. X^+ is quasi projective and $\pi_X\colon X^+ \to X^{\mathbb{G}_a}$ is affine,
2. the map $i_X\colon X^+ \to X$ is bijective on points,
3. if $X \subset \mathbb{P}(V)$ is smooth, where V is an indecomposable $\mathbb{G}_a$-representation, then $X^{\mathbb{G}_a}$ is a point and X^+ is a disjoint union of affine spaces.

We conjecture that the following analogue of BB holds for all smooth projective X:

> For every connected component X_i of X^+ the map $\pi_X\colon (X_i)_{\mathrm{red}} \to X^{\mathbb{G}_a}$ is an affine space fibration onto its image. In particular, if such an X has finitely many $\mathbb{G}_a$-fixed points, then it decomposes into a union of affine spaces.

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[BB73] A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.

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[JS19] Joachim Jelisiejew and Łukasz Sienkiewicz. Białynicki-Birula decomposition for reductive groups. *Journal de Mathématiques Pures et Appliquées.*

[JS20] Joachim Jelisiejew and Łukasz Sienkiewicz. Additive Białynicki-Birula decomposition. Work in progress.