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Some Properties of the Decompositions of Algebraic Varieties Determined by Actions of a Torus

by

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Summary. The paper is a continuation of [1]. We study here some geometric properties of the decompositions determined by actions of a one-dimensional torus on complete smooth algebraic schemes. We show that in general (even in the projective case) the decompositions are not stratifications but we are able to prove that in the projective case they satisfy some weaker conditions (useful for applications (see [2])). Finally, we find some sufficient conditions (analogous to the Smale transversality conditions [4]) under which the decompositions are in fact stratifications.

We shall use here results of [1] but we change slightly the notation introduced therein. The algebraically closed ground field is denoted by k. All algebraic schemes and morphisms (unless otherwise stated) are supposed to be defined over k. The n-dimensional vector and projective spaces (over k) will be denoted by P^n and A^n, respectively. Let T be a one-dimensional torus. In the sequel we assume that an isomorphism $T \to G_m$ is given. This enables us to identify T with G_m and $T(k)$ with $G_m(k)=k^*$. Let X be a smooth complete algebraic scheme and let an action of T on X be defined. For any $x \in X$, let $\varphi_x \colon T \to X$ be the morphism (over $k(x)$) satisfying conditions $\varphi_x(t)=tx$, for any closed $t \in T$. Then φ_x can be extended to a morphism $\bar{\varphi}_x \colon P^1 \to X$ (defined over $k(x)$). We denote $\bar{\varphi}_x(0)$ by $\lim\limits_{t \to 0} tx$, $\bar{\varphi}_x(\infty)$ by $\lim\limits_{t \to \infty} tx$ and for $a \in X^T$ we define

$$W^s(a)=\{\varphi \in X;\ \lim_{t \to 0} tx\},$$

$$W^u(a)=\{x \in X;\ \lim_{t \to \infty} tx=a\}.$$

Then $W^s(a)$, $W^u(a)$ are locally closed and they are called the stable and unstable subschemes of a, respectively (cf. [5], p. 798). The fixed point subscheme X^T can be decomposed into a union of irreducible components $X_1 \cup \ldots \cup X_r$. Define $W_i^s=\bigcup\limits_{a \in X_i} W^s(a)$, $W_i^u=\bigcup\limits_{a \in X_i} W^u(a)$, for $i=1, \ldots, r$. Then $W_i^s\,(W_i^u)$ will be called a stable (resp. unstable) subscheme of X corresponding to X_i. (Notice $W_i^s=X_i^+$, $W_i^u=X_i^-$ using the notation of [1]). It follows from [1] and [5] that $\{W_i^s\}$, $\{W_i^u\}$ are decompositions of X into locally closed subschemes. These decompositions will be called stable and unstable, respectively. The subschemes W_i^s, W_i^u will be called cells of the decompositions.

The following example shows that the closure of a cell of the given decomposition (determined as above by an action of T on X) is not in general a union of cells (even when X is a projective surface). Hence the decompositions are not in general stratifications.

Example 1. Let an action of T on $\boldsymbol{P}^2$ be given by $t\,[x_0, x_1, x_2]=[x_0, tx_1, t^2 x_2]$, for any $t \in T(k)$ and $[x_0, x_1, x_2] \in \boldsymbol{P}^2(k)$. The induced action of T on the tangent space T_{e_1} at $e_1=[0, 1, 0]$ to $\boldsymbol{P}^2$ is of the form $t\,[y_1, y_2]=[t^{-1} y_1, ty_2]$, for any $t \in T(k)$, $[y_1, y_2] \in T_{e_1}$. Let $\varphi\colon X \to \boldsymbol{P}^2$ be the blowing up of e_1. Since e_1 is fixed under the action, we have an induced action of T on X.

There exist exactly two fixed points of the action contained in $\varphi^{-1}(e_1)$ corresponding to two invariant one-dimensional subspaces of T_{e_1}. Let p_1 be the point corresponding to the subspace spanned by $[1, 0]$ and p_2 to the subspace spanned by $[0, 1]$. Then, for the action of T on X,

$$\overline{W^u(p_2)}=\overline{W^s(p_1)}=\varphi^{-1}(e_1) \quad \text{and} \quad W^u(p_1)\neq\{p_1\}.$$

Hence

$$\overline{W^u(p_2)}=W^u(p_2)\cup\{p_1\} \quad \text{and} \quad \overline{W^u(p_2)}\cap W^u(p_1)\neq\emptyset.$$

On the other hand, $\overline{W^u(p_2)}$ does not contain $W^u(p_1)$ Thus the unstable decomposition of X is not a stratification.

Definition 2. Decomposition $\{W_i^s\}$ (resp. $\{W_i^u\}$) is said to be filtrable if there exists a finite decreasing sequence $X_0 \supset X_1 \supset \ldots \supset X_m$ of closed subschemes of X such that:

(a) $X_0=X$, $X_m=\emptyset$,

(b) X_j-X_{j+1} is a cell of the decomposition $\{W_i^s\}$ (resp. $\{W_i^u\}$), for $j=0, \ldots, m-1$.

If a decomposition is a stratification then it is filtrable.

Theorem 3. *Let the algebraic variety X be projective. Then the stable and unstable decompositions are filtrable.*

Proof. According to [4] there exists an equivariant embedding of X into $\boldsymbol{P}^s$ with a linear action of T. The decompositions of $\boldsymbol{P}^s$ determined by the action are filtrable (they are even stratifications). This can be shown in the following way. We may assume that the action of T on $\boldsymbol{P}^s$ is diagonal and

$$t\,[x_0, \ldots, x_s]=[t^{n_0} x_0, \ldots, t^{n_s} x_s],$$

where $n_0, \ldots, n_s$ are integers and $n_j \leqslant n_{j+1}$, for $j=0, \ldots, s-1$. Let

$$n_0=\ldots=n_{j_1-1}<n_{j_1}=\ldots=n_{j_2-1}<n_{j_2}=\ldots<n_{j_q}=\ldots=n_s$$

and let H_i be the projective subspace of $\boldsymbol{P}^s$ defined by equations $x_0=\ldots=x_{j_i-1}=0$. Moreover, let P_i be the projective subspace of $\boldsymbol{P}^s$ defined by equations $x_0=\ldots=x_{j_i-1}=x_{j_{i+1}}=\ldots=x_s=0$, for $i=0, \ldots, q$. Then

$$\bigcup P_j=(\boldsymbol{P}^s)^T, \qquad H_i \supset H_{i+1}, \quad H_0=\boldsymbol{P}^s, \qquad H_{q+1}=\emptyset$$

and the difference $H_i - H_{i+1}$ is the cell of the stable decomposition (of P^s) composed of those points x that $\lim_{t\to 0} tx \in P_i$.

In order to show that the stable decomposition $\{W_i^s\}$ is filtrable notice first that

$$X^T = (P^s)^T \cap X = \bigcup P_j \cap X \quad \text{and} \quad P_j \cap P_{j'} = \emptyset$$

for $j \neq j'$. Hence irreducible components of $P_j \cap X$, for $j = 1, \ldots, q$, coincide with irreducible components of X^T. Moreover, the intersection $(H_i - H_{i+1}) \cap X$ is composed of all such points $x \in X$ that $\lim_{t\to\infty} tx \in P_i \cap X$. Hence $(H_i \cap X) - (H_{i+1} \cap X)$ is a union of some cells of the stable decomposition, say

$$(H_i \cap X) - (H_{i+1} \cap X) = W_{i_1}^s \cup \ldots \cup W_{i_l}^s.$$

Let $\pi_i \colon (H_i - H_{i+1}) \to P_i$ be the fibration of the cell $H_i - H_{i+1}$ determined by the action of T on P^s. Then

$$W_{i_j}^s \subset \pi_i^{-1}(W_{i_j}^s \cap P_i).$$

Since

$$(W_{i_j}^s \cap P_i) \cap (W_{i_k}^s \cap P_i) = \emptyset \quad \text{for} \quad j \neq k,$$

and $W_{i_j}^s \cap P_i$ is closed (as an irreducible component of X^T), the intersection $\overline{W_{i_j}^s} \cap \cap \overline{W_{i_k}^s}$, for $j \neq k$, is contained in $H_{i+1} \cap X$. Therefore the union

$$H_{i+1} \cap X \cup W_{i_1}^s \cup \ldots \cup W_{i_r}^s$$

is closed, for $r = 1, \ldots, l$.

Suppose that we have already defined a sequence $X_0 \supset \ldots \supset X_p$ of closed subschemes of X such that $X_0 = X$, $X_p = H_i \cap X$ and $X_j - X_{j+1}$ is a cell of the stable decomposition, for $j = 0, \ldots, p-1$. Then we put

$$X_{p+j} = (H_{i+1} \cap X) \cup W_{i_1}^s \cup \ldots \cup W_{i_{l-j}}^s, \quad \text{for} \quad j = 1, \ldots, l.$$

This proves that the stable decomposition $\{W_i^s\}$ is filtrable. The same result also holds for the unstable decomposition.

R e m a r k. In fact we have proved above that the decompositions of a projective space are filtrable and that the property of the action that the stable and unstable decompositions are filtrable is hereditary (with respect to equivariant inclusions). The author is not able to prove that Theorem 3 holds when projectivity of X is replaced by the weaker assumption of completeness.

DEFINITION 4. We say that the action of T on X satisfies the transversality condition (of Smale) if any cell W_i^s intersects any cell W_j^u transversally (i.e. if for any closed point $x \in W_i^s \cap W_j^u$, the tangent space to X at x is spanned by the tangent spaces to W_i^s and W_j^u at x).

THEOREM 5. *Let X be complete and suppose that X^T is finite and the action of T on X satisfies the transversality condition. Then stable and unstable decompositions are stratifications.*

In the sequel we shall always assume that X^T is finite and the transversality condition holds for the action. First we fix some notation and prove some results which play an auxiliary role in the proof of Theorem 5, but may have an independent value.

It follows from [1, 6] that for any $a \in X^T$ there exists a T-invariant etale morphism ψ of a T-invariant neighbourhood U of a into A^n (with a diagonal action of T) which maps a into the origin of A^n. In some coordinate system of A^n the action of T on A^n is given by

$$t\,[x_1, \ldots, x_n] = [t^{s_1} x_0, \ldots, t^{s_m} x_m, t^{s_{m+1}} x_{m+1}, \ldots, t^{s_n} x_n],$$

where $s_1, \ldots, s_m$ are negative and $s_{m+1}, \ldots, s_n$ positive integers, $m = \dim W^u(a)$, $n-m = \dim W^s(a)$. Suppose that $b \in X^T$ and that $W^s(a) \cap W^u(b) \neq \emptyset$. Let π_0 be the restriction to $\psi(W^u(b))$ of the projection of A^n onto the product of the first m axis.

LEMMA 6. *π_0 is dominant.*

Proof. Since $W^s(a)$, $W^u(b)$ are smooth and intersect transversally at any $c \in W^s(a) \cap W^u(b)$ and since ψ is etale, $\psi(W^s(a))$, $\psi(W^u(b))$ are also smooth and intersect transversally at $\psi(c)$. Hence the map $d\pi_0$ of the tangent space of $\psi(W^u(b))$ at $\psi(c)$ into the tangent space of A^m (equal to the product of the first m axis) at the origin induced by π_0 is surjective. Thus $\dim \pi_0\big(\psi(W^s(b))\big) = m$ and π_0 is dominant.

LEMMA 7. *Let a, b be points of X^T. If*

$$W^s(a) \cap W^u(b) \neq \emptyset, \quad \text{then} \quad \overline{W^u(b)} \supset W^u(a).$$

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$$W^u(a) \cap W^s(b) \neq \emptyset, \quad \text{then} \quad \overline{W^s(b)} \supset W^s(a).$$

Proof. Suppose that

$$W^s(a) \cap W^u(b) \neq \emptyset.$$

Let ψ, π_0 be as above. Let c be any closed point of $W^s(a) \cap W^u(b)$. Let $\mathcal{O}$ be the local ring of $\psi(W^u(b))$ at $\psi(c)$. Since $\psi(W^u(b))$ is smooth, $\mathcal{O}$ is regular. Let us fix an isomorphism

$$\iota\colon A^n \to \operatorname{Spec} k\,[X_1, \ldots, X_n]$$

such that $\psi(a)$ is the origin 0 of $\operatorname{Spec} k\,[X_1, \ldots, X_n]$. We shall identify (geometric) points of A_n and $\operatorname{Spec} k\,[X_1, \ldots, X_n]$ corresponding under ι. The inclusion $\psi(W^u(b)) \subset A^n$ induces a homomorphism $\zeta\colon k\,[X_1, \ldots, X_n] \to \mathcal{O}$. Let $(c_1, \ldots, c_n) \in k^n$ be the coordinates of $\psi(c)$. Then $c_1 = \ldots = c_m = 0$ (since $\psi(c) \in W^s(0)$) and we may choose $X_{i_1}, \ldots, X_{i_l}$, where $l = \dim W^u(b)$, such that

$$k\,[[\zeta(X_{i_1} - c_{i_1}), \ldots, \zeta(X_{i_l} - c_{i_l})]] = \hat{\mathcal{O}};$$

moreover, it follows from Lemma 6 that we may assume that $i_1=1, \ldots, i_m=m$. There exists a homomorphism $\alpha\colon \hat{\mathcal{O}} \to k[[u, Y_{i_1}, \ldots, Y_{i_l}]]$, where $u, Y_{i_1}, \ldots, Y_{i_l}$ are indeterminates such that, for $j=1, \ldots, l$,

$$\alpha\left(\zeta\left(X_{i_j}-c_{i_j}\right)\right)=\begin{cases} u^{-s_{i_j}} Y_{i_j} & \text{if} \quad s_{i_j}<0, \\ Y_{i_j} & \text{if} \quad s_{i_j}>0. \end{cases}$$

The homomorphism α determines a $k[[u, Y_{i_1}, \ldots, Y_{i_l}]]$-rational point of $\psi(W^u(b)) \subset A^n$ with coordinates

$$(\bar{c}_1, \ldots, \bar{c}_n) \quad \text{and} \quad \bar{c}_1=u^{-s_1} Y_1, \ldots, c_m=u^{-s_m} Y_m .$$

Now, the action of the geometric point u of T on the point with coordinates $(\bar{c}_1, \ldots$ $\ldots, \bar{c}_n)$ gives the point with coordinates

$$(u^{s_1} \bar{c}_1, \ldots, u^{s_n} \bar{c}_n)=(Y_1, \ldots, Y_m, u^{s_{m+1}} c_{m+1}, \ldots, u^{s_n} c_n)$$

belonging to

$$\psi(W^u(b))(k[[u, Y_{i_1}, \ldots, Y_{i_l}]]).$$

Let κ be the canonical morphism of $k[[u, Y_{i_1}, \ldots, Y_{i_l}]]$ onto

$$k[[u, Y_{i_1}, \ldots, Y_{i_l}]]/(u)=k[[Y_{i_1}, \ldots Y_{i_l}]].$$

Then the image of

$$(Y_1, \ldots, Y_m, u^{s_{m+1}} c_{m+1}, \ldots, u^{s_n} c_n)$$

under κ has coordinates $(Y_1, \ldots, Y_m, 0, \ldots, 0)$ and the image is a geometric point of $\overline{\psi(W^u(b))}$. Since $Y_1, \ldots, Y_m$ are indeterminates,

$$\psi(W^u(a))=W^u(0) \in \overline{\psi(W^u(b))}.$$

Thus $W^u(a) \subset \overline{W^u(b)}$.

The second part of the lemma follows immediately from the first one.

COROLLARY 8. *Under the assumption of Lemma 7, if $a \neq b$ and $W^s(a) \cap W^u(b) \neq \emptyset$, then* $\dim W^u(b) > \dim W^u(a)$; *if $a \neq b$ and $W^u(a) \cap W^s(b) \neq \emptyset$ then* $\dim W^s(b) >$ $> \dim W^s(a)$.

LEMMA 9. *Let $a, b \in X^T$, $a \neq b$ and $a \in \overline{W^s(b)} \cap X^T$. Then*

$$\overline{W^s(b)} \cap W^u(a) \neq \{a\}.$$

Proof. Suppose that

$$\overline{W^s(b)} \cap W^u(a)=\{a\}.$$

Then

$$\psi(W^s(b) \cap U) \cap \psi(W^u(a))=\{\psi(a)\}$$

(since ψ is $1-1$ on $W^u(a)$ and $\psi^{-1}\psi(W^u(a))=W^u(a)$). As in the proof of Lemma 7 we shall identify A^n and $\operatorname{Spec} k[X_1, \ldots, X_n]$. Let $f_1=0, \ldots, f_s=0$ be a set of equations of $\psi(\overline{W^s(b)})$ in A^n. We may assume that the polynomials $f_1, \ldots, f_s$ are semiinvariant under the induced action of T on $k[X_1, \ldots, X_n]$. Since $\psi(a)=(0, \ldots, 0)$, it follows from our assumption that $(0, \ldots, 0)$ is the only solution of the system

$$f_1=0, \ldots, f_s=0, \quad X_{m+1}=0, \ldots, X_n=0 .$$

Therefore

$$X_i^r \in (f_1, \ldots, f_s, X_{m+1}, \ldots, X_n)$$

for some integer r, where $i=1, \ldots, m$, i.e., there exists

$$g_{ij}, h_{ij} \in k\,[X_1, \ldots, X_n]$$

such that

$$X_i^r = \sum_{j=1}^{s} f_j\, g_{ij} - \sum_{j-1}^{n-m} X_{m+j}\, h_{ij}, \qquad i=1, \ldots, m.$$

We may assume that all summands $f_j\, g_{ij}$, $X_{m+j}\, h_{ij}$ are semiinvariants of the same weight as X_i^r. Hence h_{ij} have to be of negative weights, in particular they have to belong to the ideal $(X_1, \ldots, X_m)$. Let Y be the subscheme of A^n defined by the ideal

$$i=(X_1^r+\Sigma X_{m+j}\, h_{1j}, \ldots, X_m^r+\Sigma X_{m+j}\, h_{mj}).$$

Then $Y \supset \psi\big(W^s(b)\cap U\big)\cup W^s(a)$ (because the ideal generated by $X_i^r+\Sigma X_{m+j}\, h_{ij}$, for $i=1, \ldots, m$, is contained in the ideals $(X_1, \ldots, X_m)$ and $(f_1, \ldots, f_s)$).

We shall prove first that the tangent cone $C\,(Y, 0)$ to Y at the origin $0=(0, \ldots, 0)$ is contained in $W^s(0)$. In order to show this it suffices to prove, that for $i=1, \ldots, m$, there exist an integer r_i and a polynomial $F_i \in I$ such that $F_i = X_i^{r_i}$+terms of degree greater than r_i. Let $1 \leqslant i \leqslant m$,

$$r_i' = \left[\frac{\sum_{j=1}^{m} s_j}{s_i}\right]+1, \quad r_i = r\cdot r_i'$$

and consider

$$F_i^{(0)}=(X_i^r+\Sigma X_{m+j}\, h_{ij})^{r_i'}.$$

Let $aX_1^{i_1}\ldots X_n^{i_n}$ be any monomial occurring in $F_i^{(0)}$ with a nonzero coefficient a.

Since $F_i^{(0)}$ is semi-invariant of (negative) weight $r_i\, s_i$ and $r_i\,|s_i|>(r-1)\sum_{j=1}^{m}|s_j|$, there exists an integer k, $1 \leqslant k \leqslant m$, such that $i_k \geqslant r$. Choose such k and replace in $aX_1^{i_1}\ldots X_n^{i_n}$ the factor $X_k^{i_k}$ by

$$X_k^{i_k-r}\cdot\left(\sum_{j=1}^{n-m} X_{m+j}\, h_{kj}\right).$$

Then $aX_1^{i_1}\cdot\ldots\cdot X_n^{i_n}$ and the obtained polynomial are congruent modulo I. Next make such substitutions for all monomials occurring in $F_i^{(0)}-X_i^{r_i}=\Sigma X_{m+j}\, h_{ij}^{(0)}$. Then we obtain a polynomial $\Sigma X_{m+j}\, h_{ij}^{(1)}$ congruent to $\Sigma X_{m+j}\, h_{ij}^{(0)}$ modulo I and such that the degree of any monomial occurring in $\Sigma X_{m+j}\, h_{ij}^{(1)}$ in the variables $X_{m+1}, \ldots, X_n$ is greater than 1. Moreover, $\Sigma X_{m+j}\, h_{ij}^{(1)}$ is semi-invariant of weight $r_i\, s_i$. Hence we may proceed as above. After r_i steps we get a polynomial $\Sigma X_{m+j}\, h_{ij}^{(r_i)}$ congruent to $\Sigma X_{m+j}\, h_{ij}$ modulo I and such that the degree of any monomial occurring in $\Sigma X_{m+j}\, h_{ij}^{(r_i)}$ with respect to the variables $X_{m+1}, \ldots, X_n$ is greater than r_i. Hence any monomial occurring in $\Sigma X_{m+j}\, h_{ij}^{(r_i)}$ is of degree greater than r_i and the polynomial

$$F_i = X_i^{r_i}+\Sigma X_{m+j}\, h_{ij}^{(r_i)}$$

satisfies the desired conditions.

Since $C(Y, 0) \subset W^s(0)$, the tangent space $T(Y, 0)$ to Y at 0 is also contained in $W^s(0)$. By corollary 2 [3], the union Y_0 of all irreducible components of Y containing 0 is contained in $W^s(0)$. Thus

$$\psi(W^s(b) \cap U) \subset Y_0 \subset W^s(0) = \psi(W^s(a)) \quad \text{and} \quad U \cap W^s(b) \subset W^s(a).$$

Hence $a = b$, a contradiction.

Corollary 10. *Let $a_1, \ldots, a_p$ be a finite sequence of closed points of X^T such that $a_i \neq a_{i+1}$ and*

$$W^s(a_i) \cap W^u(a_{i+1}) \neq \emptyset, \quad \text{for} \quad i = 1, \ldots, p-1.$$

Then

$$p \leqslant \dim X + 1.$$

Proof. It follows from Lemma 7 that $\overline{W^s(a_i)} \supset W^s(a_{i+1})$, for $i = 1, \ldots, p-1$. Since $a_i \neq a_{i+1}$,

$$W^s(a_i) \neq W^s(a_{i+1}) \quad \text{and} \quad \overline{W^s(a_i)} \neq \overline{W^s(a_{i+1})}, \quad \text{for} \quad i = 1, \ldots, p-1.$$

Thus $p \leqslant \dim X + 1$.

Corollary 11. *Let a, b be closed points of X^T. Then $a \in \overline{W^s(b)}$ iff there exists a finite sequence of closed points $a_1, \ldots, a_p \in X^T$ satisfying the following conditions:*

(1) $$a_1 = a, \quad a_p = b,$$

(2) $$W^s(a_{i+1}) \cap W^u(a_i) \neq \emptyset.$$

Proof. Suppose that $a \in \overline{W^s(b)}$. Let $a_1 = a$. If $b \neq a$ then by Lemma 9 there exists a closed point $x_1 \notin X^T$ belonging to $\overline{W^s(b)} \cap W^u(a)$. Let $a_2 = \lim_{t \to 0} tx_1$. Then

$$a_2 \in \overline{W^s(b)} \quad \text{and} \quad W^s(a_2) \cap W^u(a_1) \neq \emptyset.$$

Hence we may repeat the argument replacing a_1 by a_2. If $a_2 = b$, then we find x_2 and a_3 such that

$$a_3 \in \overline{W^s(b)} \quad \text{and} \quad x_2 \in W^s(a_3) \cap W^u(a_2) \neq \emptyset.$$

By Corollary 10 after at most $n+1$ steps this procedure must stop. Hence there exists an integer $p \leqslant n$ such that $a_p = b$. The sequence $a_1, \ldots, a_p$ satisfies conditions (1) and (2).

Conversely, if $a_1, \ldots, a_p$ satisfy (1) and (2), then it follows from Lemma 7 that

$$W^s(a_i) \subset \overline{W^s(a_{i+1})} \quad \text{for} \quad i = 1, \ldots, p-1.$$

Hence

$$a \in W^s(a) = W^s(a_1) \subset \overline{W^s(a_p)} = \overline{W^s(b)}.$$

Proof of Theorem 5. Let b a closed point of X^T. Let c be a closed point of $\overline{W^s(b)}$ and let $a = \lim_{t \to 0} tc$. We shall show that $W^s(a) \subset \overline{W^s(b)}$. Clearly $a \in \overline{W^s(b)}$. Let $a_1, \ldots, a_p$ and $x_1, \ldots, x_p$ be as in the proof of Corollary 11. If $p = 1$ then $a = b$

and the inclusion $W^s(b) \subset \overline{W^s(a)}$ is clear. Suppose that the inclusion $W^s(b) \subset \overline{W^s(a)}$ holds if $p \leqslant k$ and let $p = k+1$. It follows from our assumptions that $W^s(a_2) \subset \overline{W^s(b)}$. Since $x_1 \in W^s(a_2) \cap W^u(a_1)$, by Lemma 7 $W^s(a_1) \subset \overline{W^s(a_2)}$. Thus

$$W^s(a) = W^s(a_1) \subset \overline{W^s(a_2)} \subset \overline{W^s(b)}$$

and the proof is completed.

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А. Бялыницки-Бируля, **Некоторые свойства алгебраических декомпозиции варнантоэ определенных действием торуса**

Содержание. Настоящая работа является продолжением [1]. Исследуются геометрические свойства декомпозиции определенной действием одномерного торуса на полной гладкой алгебраической схеме.