Instruments of algebraic torus action

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headline

- *Can one hear the shape of a drum?* [Katz, *AMM* 1966]
- Can one identify a (complex, projective) manifold by knowing the eigenvalues of an action of an algebraic torus?

motivation: LeBrun–Salamon conjecture

- The conjecture in Riemannian differential geometry: positive quaternion-Kähler manifolds are symmetric spaces (Wolf spaces).
- The conjecture in complex algebraic geometry: every Fano complex contact manifold is homogeneous and in fact the projectivisation of the minimal adjoint orbit of a simple group.

motivation: Fano contact manifolds

- Let L be an ample line bundle on a complex manifold X, $\dim X = 2n+1$, a contact form $\theta \in \mathrm{H}^0(X, \Omega_X \otimes L)$ is such that $(d\theta)^{\wedge n} \wedge \theta$ nowhere vanishes; this implies $-K_X = (n+1)L$
- Let F be the kernel of $\theta : TX \to L$ then $d\theta$ defines nondegenerate skew-symmetric pairing:

$$d\theta : F \times F \to L$$

- Partial results on contact and quaternion-Kähler manifolds: small $\dim X$, big torus, L has many sections, [Hitchin, Poon, LeBrun, Salamon, Herrera^2, Bielawski, Fang, Druel, Beauville]
- We may assume $\operatorname{Pic} X = \mathbb{Z} \cdot L$, otherwise

$$(X, L) = (\mathbb{P}^{2n+1}, \mathcal{O}(2)) \ \text{ or } \ (\mathbb{P}(T\mathbb{P}^{n+1}), \mathcal{O}(1))$$

what everyone knows about toric manifolds

Notation: H will denote an algebraic torus with M the lattice of characters; r denotes the rank of H and M.

If X is a toric manifold and L ample line bundle on X then we get a lattice polytope $\Delta = \Delta(X, H, L, \mu)$ in M such that

$$\mathrm{H}^0(X, L) = \bigoplus_{u \in M \cap \Delta} \mathbb{C}u$$

- fixed points of action of H are associated to vertices of Δ
- Δ is a simple polytope and for every vertex u of Δ the cone $\mathbb{R}_{\geq 0} \cdot (\Delta - u)$ is regular in M.

polytope of section

Let H act effectively* on (X, L), with given linearization $\mu : H \times L \to L$

- We have decomposition of space of sections into eigenspaces

$$\mathrm{H}^0(X, L) = \bigoplus_{u \in M} \mathrm{H}^0(X, L)_u$$

- By $\widetilde{\Gamma}(X, H, L, \mu) \subset M$ we denote the eigenvalues of the action of H on $\mathrm{H}^0(X, L)$ and by $\Gamma = \Gamma(X, H, L, \mu)$ their convex hull in $M_{\mathbb{R}}$.

polytope of fixed points

- We have decomposition of the set of fixed points

$$X^H = Y_1 \sqcup \cdots \sqcup Y_s$$

- By $\widetilde{\Delta}(X, H, L, \mu) \subset M$ we denote the set of the characters $\mu(Y_i)$ of the action of H on Y_i's and by $\Delta = \Delta(X, H, L, \mu)$ their convex hull in $M_{\mathbb{R}}$.
- A connected component $Y \subset X^H$ is called *extremal* if $\mu(Y)$ is a vertex of Δ.

first observation

- $\Delta(L^{\otimes m}) = m \cdot \Delta(L)$ and $\Gamma(L^{\otimes m}) \supseteq m \cdot \Gamma(L)$
- $\Delta(L) = \Gamma(L)$ if L is base point free
- hence $\Gamma(L) \subseteq \Delta(L)$

Note: if $\Delta(L)$ is “small” then there should not be too many fixed points components.

the compass

Let $Y \subset X^H$ be a connected component.
Take $y \in Y$ and consider the action of H on $T^*_y X$: it splits into eigenspaces associated to some characters in M; the trivial eigenspace is $T^*_y Y$

The set non-zero (multiple) characters of this action we call *the compass* of the action of H on the component Y and we denote it $\mathcal{C}(Y, X, H)$

Fact: the elements of the compass generate the semigroup

$$(\mathbb{R}_{\geq 0} \cdot (\Delta(X, L, H, \mu) - \mu(Y))) \cap M'$$

where $M' \subseteq M$ is the lattice of characters of $H' = (H/\mathrm{stabilizer})$, a quotient of H which acts acts effectively on X.

reduction of the action, 1

Consider a sequence of tori

$$0 \longrightarrow H_1 \xrightarrow{\pi} H \xrightarrow{\iota} H_2 \longrightarrow 0$$

and the associated sequence of lattices of characters

$$0 \longrightarrow M_2 \xrightarrow{\iota} M \xrightarrow{\pi} M_1 \longrightarrow 0$$

We have the action of H_2 on components of X^{H_1} and for every connected component $Y_1 \subset X^{H_1}$ we get

$$Y_1^{H_2} = X^H \cap Y_1$$

For a general choice $H_1 \hookrightarrow H$ we have $X^{H_1} = X^H$

reduction of the action, 2

The restriction of the action to $H_1 \hookrightarrow H$ implies

$$\pi(\Delta(X, L, H, \mu)) = \Delta(X, L, H_1, \mu_{H_1})$$

in particular extremal fixed point components of X^H map into extremal fixed point components of X^{H_1}.

For every pair of connected components $Y_1 \subset X^{H_1}$ and $Y \subset Y_1^{H_2}$ we have

- the elements of $\mathcal{C}(Y_1, X, H_1)$ are π-projections of elements from $\mathcal{C}(Y, X, H)$
- the elements of $\mathcal{C}(Y, Y_1, H_2)$ are those in $\mathcal{C}(Y, X, H)$ which are in the kernel of π

example 1: odd quadrics, 1

The torus $H = (\mathbb{C}^*)^n$ with coordinates $(t_1, \ldots, t_n)$ acts on $\mathbb{C}^{2n+1}$

$$(t_1, \ldots, t_n) \cdot (z_0, z_1, z_2, \ldots, z_{2n-1}, z_{2n}) = \\ (z_0, t_1 z_1, t_1^{-1} z_2, \ldots, t_n z_{2n-1}, t_n^{-1} z_{2n})$$

The action of H descends to an effective action on the quadric $\mathbb{Q}^{2n-1} \subset \mathbb{P}^{2n}$ given by equation

$$z_0^2 + z_1 z_2 + \cdots + z_{2n-1} z_{2n} = 0$$

with $2n$ isolated fixed points:

$$[0,1,0,\ldots,0,0],\ [0,0,1,\ldots,0,0,],\ldots, \\ \ldots[0,0,0,\ldots,1,0],\ [0,0,0,\ldots,0,1]$$

example 1: odd quadrics, 2

Let M be the lattice of characters of H with the basis of $\mathbb{Z}^n$

$$e_1 = (1, 0, \dots, 0, 0), \dots, e_n = (0, 0, \dots, 0, 1)$$

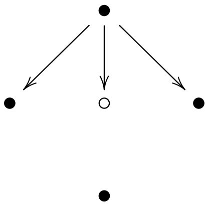
Then

$$\Delta(\mathbb{Q}^{2n-1}, \mathcal{O}(1), H) = \mathrm{conv}(\pm e_1, \dots, \pm e_n)$$

and the compass of H at the the fixed point associated to the character e_i consists of $-e_i$ and $\pm e_j - e_i$, for $j \neq i$. Note that the compass generates $\mathbb{R}_{\geq 0}(\Delta - e_i) \cap M$

3-dimensional quadric

Two-dimensional torus acting on the 3-dimensional quadric:
four fixed points, five sections, three elements in the compass:



example 2: even quadrics, 1

The torus $H = (\mathbb{C}^*)^n$ with coordinates $(t_1, \ldots, t_n)$ acts on $\mathbb{C}^{2n}$

$$(t_1, \ldots, t_n) \cdot (z_1, z_2, \ldots, z_{2n-1}, z_{2n}) = (t_1 z_1, t_1^{-1} z_2, \ldots, t_n z_{2n-1}, t_n^{-1} z_{2n})$$

The action of H descends to an action of the quotient torus $H' = H/\langle(-1, \ldots, -1)\rangle$ on the quadric $\mathbb{Q}^{2n-2} \subset \mathbb{P}^{2n-1}$ given by equation

$$z_1 z_2 + \cdots + z_{2n-1} z_{2n} = 0$$

with $2n$ isolated fixed points:

$$[1, 0, \ldots, 0, 0],\ [0, 1, \ldots, 0, 0,], \ldots, \ldots [0, 0, \ldots, 1, 0],\ [0, 0, \ldots, 0, 1]$$

example 2: even quadric, 2

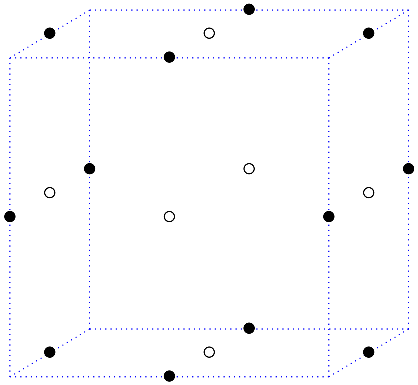
As before, $M = \mathbb{Z}^n$ generated by e_i's and $M' \subset M$ index 2 sublattice of vectors $\sum_i a_i e_i$ such that $\sum_i a_i$ is even.
As before

$$\Delta(\mathbb{Q}^{2n-2}, \mathcal{O}(1), H) = \mathrm{conv}(\pm e_1, \ldots, \pm e_n)$$

Now the compass of H at the fixed point associated to the character e_i consists of $\pm e_j - e_i$, for $j \neq i$. Note that the compass generates $\mathbb{R}_{\geq 0}(\Delta - e_i) \cap M'$

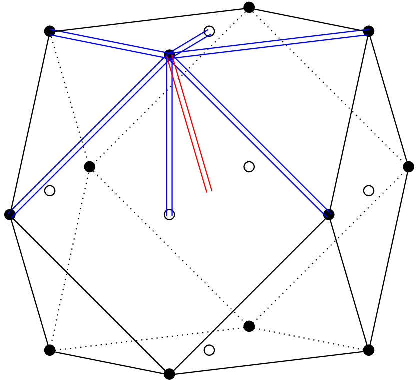
example 3: minimal nilpotent orbit of B_3

B_3 root system



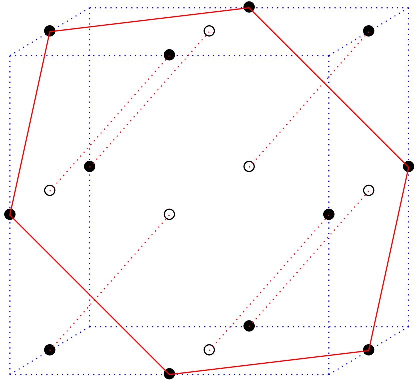
example 3: minimal nilpotent orbit of B_3

Root polytope of B_3 and the compass.



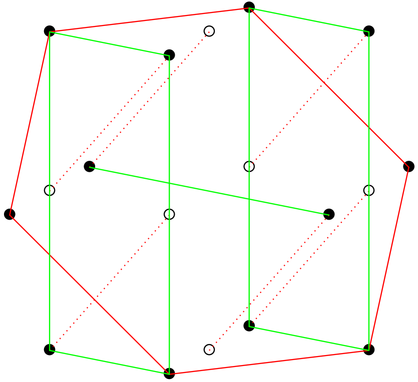
example 3: minimal nilpotent orbit of B_3

Downgrading the action.



example 3: minimal nilpotent orbit of B_3

Downgrading and restricting the action



BB decomposition

For $H = \mathbb{C}^*$ with coordinate t and X projective manifold we have Białynicki-Birula decomposition:

- Take decomposition $X^H = Y_1 \sqcup \cdots \sqcup Y_s$ and for every Y_i by $\nu^{\pm}(Y_i)$ denote the positive and negative number of characters in the compass.
- Define
  $X_i^+ = \{x \in X : \lim_{t\to 0} t \cdot x \in Y_i\}$
  $X_i^- = \{x \in X : \lim_{t\to\infty} t \cdot x \in Y_i\}$
- Then
  - $X = X_1^+ \sqcup \cdots \sqcup X_s^+ = X_1^- \sqcup \cdots \sqcup X_s^-$,
  - partial order $Y_i \prec Y_j \Leftrightarrow \overline{X_i^+} \supset Y_j$ agrees with $\mu(Y_i) < \mu(Y_j)$
  - the unique dense $^{\pm}$-component is called source/sink,
  - $X_i^{\pm} \to Y_i$ is a $\mathbb{C}^{\nu^{\pm}(Y_i)}$ fibration,
  - $H_m(X,\mathbb{Z}) = \bigoplus_i H_{m-2\nu^+(Y_i)}(Y_i,\mathbb{Z}) = \bigoplus_i H_{m-2\nu^-(Y_i)}(Y_i,\mathbb{Z})$

BB decomposition – consequences

Assume in addition that $\operatorname{Pic} X = \mathbb{Z} \cdot L$ and $Y_0 \subset X^H$ is the source. Then X Fano and one of the following holds:

1. $\dim Y_0 > 0$ and
   - Y_0 is Fano with $\operatorname{Pic} Y_0 = \mathbb{Z} \cdot L$,
   - the complement of X_0^+ is of codimension ≥ 2,
   - $\mathrm{H}^0(X, L) \to \mathrm{H}^0(Y_0, L)$ is surjective.
2. Y_0 is a point and
   - X_0^+ is an affine space
   - $D = X \setminus X_0^+$ is an irreducible divisor in the system $|L|$,
   - there exists the unique fixed point component $Y_1 \subset X^H$ such that $\mu(Y_1)$ is minimal in $\widetilde{\Delta}(X, L, H, \mu) \setminus \mu(Y_0)$,
   - X_1^+ associated to Y_1 is dense in D.

BB-decomposition – case $\mathrm{rk}(H) \geq 1$

- Extremal fixed point components are in bijection with vertices of $\Delta(X, L, H)$.
- If Pic $X = \mathbb{Z}$ and $r \geq \dim X - 4$ then

$$\Delta(X, L, H) = \Gamma(X, L, H)$$

Thus knowing the weights of sections of L we can try to recover the set of fixed point components

$$\widetilde{\Gamma}(X, L, H) \rightsquigarrow \widetilde{\Delta}(X, L, H)$$

from fixed points to sections

Grothendieck-Atiyah-Bott-Berline-Vergne localization in cohomology and Riemann-Roch theorem (simplest version):

Assume that X^H consists of isolated points $y_1, y_2, \dots y_k$. Take $\mu_i = \mu(y_i)$ and $\nu_{i,j}$ are elements of $\mathcal{C}(y_i, X, H)$.
Then the character of the representation of H on $\mathrm{H}^0(X, L^{\otimes m})$ is equal

$$\sum_{i=1}^{k} \frac{t^{m\mu_i}}{\prod_j (1 - t^{\nu_{i,j}})}$$

Corollary. Suppose that a simple group G with a maximal torus H acts on X, $\operatorname{Pic} X = \mathbb{Z}L$, so that the data $\mu_i,\ \nu_{i,j}$ is the same as for a G-homogeneous manifold $\widehat{X}$, $\operatorname{Pic} \widehat{X} = \mathbb{Z}\widehat{L}$. Then

$$(X, L) = (\widehat{X}, \widehat{L})$$

back to contact and quaternion-Kähler manifolds

Theorem. Let X be a contact Fano manifold of dimension $2n+1$, with $n \geq 3$, whose group of contact automorphisms G is reductive and contains an algebraic torus H of rank $\geq n-2$. Then X is homogeneous.

Theorem. Let M be a positive quaternionic Kähler manifold of dimension $4m$. If $m \leq 4$ then M is a Wolf space.

proof: main ideas

Use sequence

$$0 \longrightarrow F \longrightarrow TX \xrightarrow{\theta} L \longrightarrow 0$$

to get the linearization μ of G acting on L with adjoint action on $\mathrm{H}^0(X, L) = \mathfrak{g}$.
Pairing $d\theta : F \times F \to L$ defines symmetry in the compass at every fixed point component Y:

- after renumbering $\nu_0 = -\mu(Y)$ and $\nu_i + \nu_{i+n} = \nu_0$,
- if $\mu(Y) \neq 0$ then Y is isotropic, dim $Y + 1$ equals multiplicity of $-\mu(Y)$ in the compass,
- if $\mu(Y) = 0$ then Y is contact, hence of odd dimension.

proof: main steps

1. $\Delta(X, L, H, \mu) = \Gamma(X, L, H, \mu)$, because extremal fixed point components are isotropic
2. G is semisimple (no torus component), because Δ is of maximal dimension
3. G is simple (not a product), because otherwise Δ is a coproduct of root polytopes
4. Analyse root polytopes for simple groups in lattices of weights. Case-by-case analysis, use information about root systems.

proof: adjoint orbits of simple groups

Dimension of the adjoint orbit $=$ minimal number of generators of the cone semingroup $+1$:

A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$2r-1$	$4r-5$	$2r-1$	$4r-7$	21	33	57	15	5

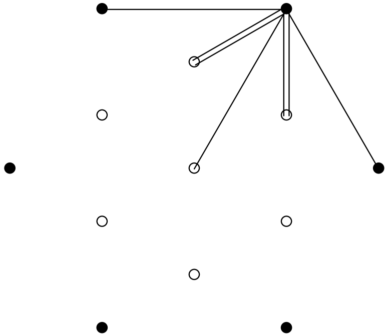
Moreover in case A_r we have Pic $= \mathbb{Z}^2$ and in case C_r the line bundle L is divisible by 2 in Pic.

Because of the Weyl grop action the compass satisfies symmetry.

As the result: careful discussion needed for A_2/G_2, B_3, D_4.

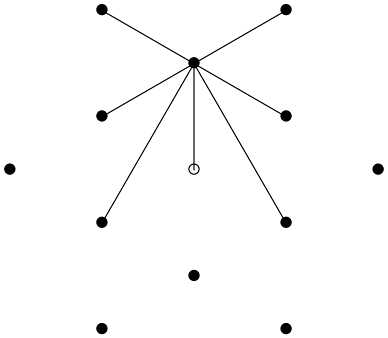
example: case of A_2/G_2, $\dim X = 7$

Discussion: compass at extremal fixed points.



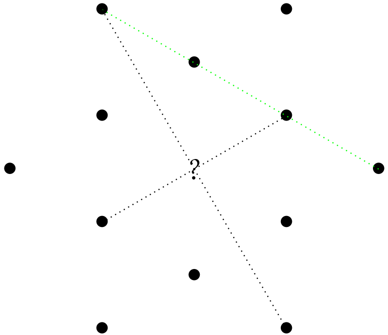
example: case of A_2/G_2, $\dim X = 7$

Discussion: compass at "inner" fixed points.



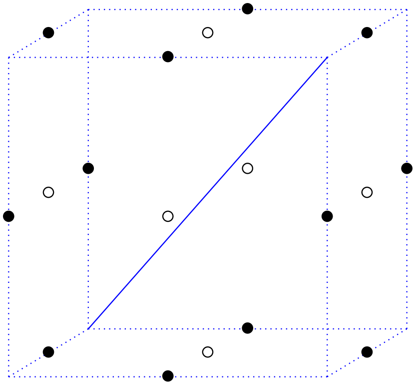
example: case of A_2/G_2, $\dim X = 7$

Discussion: no fixed point at 0, inner points are single



example: case of A_2/G_2, $\dim X = 7$

Conclusion: the case of A_2/G_2, $\dim X = 7$, is projection of the system associated to root type B_3; projection of the system B_3.



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