RIGIDITY OF MORI CONE FOR FANO MANIFOLDS

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Abstract. Mori cone is rigid in smooth connected families of Fano manifolds.

The aim of this note is to give a positive answer to a question raised at a workshop *Rational curves on Algebraic Varieties* organized in American Institute of Mathematics, Palo Alto, CA, in May 2007, [Alt07, Question 0.7].

We consider algebraic varieties defined over the field of complex numbers $\mathbb{C}$. Manifolds are smooth connected varieties.

If X is a complex projective manifold then by $N^1(X) \subset H^2(X,\mathbb{R})$ and $N_1(X) \subset H_2(X,\mathbb{R})$ we denote $\mathbb{R}$-linear subspaces spanned by cohomology and homology classes of, respectively, Cartier divisors and algebraic curves on X. The cone of curves, or Mori cone of X, $\mathcal{C}(X) \subset N_1(X)$ and the cone of nef divisors $\mathcal{P}(X) \subset N^1(X)$ are $\mathbb{R}^*_{>0}$-spanned by, respectively, the classes of curves, or effective 1-cycles, and numerically effective divisors, hence $\mathcal{P}(X) := \{\chi \in N^1(X) : \forall\alpha \in \mathcal{C}(X) \ \ \chi\cdot\alpha \geq 0\}$. That is, $\mathcal{P}(X) = \mathcal{C}(X)^\vee$ in the sense of the intersection pairing of $N^1(X)$ and $N_1(X)$. See [Mor82] for more information on these objects.

A manifold X is Fano if its anticanonical divisor $-K_X$ is ample. If X is Fano then $H^1(\mathcal{O}_X) = 0$ hence $N^1(X) = H^2(X,\mathbb{R})$ and $N_1(X) = H_2(X,\mathbb{R})$.

A smooth family of projective manifolds $\pi : X \to S$ is, by definition, a smooth projective morphism of connected complex manifolds with connected fibers. Geometric fibers of π are denoted by $X_s = \pi^{-1}(s)$ for $s \in S$. By a known result in differential geometry any such family is topologically locally trivial in complex topology. A local topological trivialization of π induces identification of homology and cohomology of neighboring fibers.

Theorem 1. *Let us assume that $\pi : X \to S$ is a smooth family of Fano manifolds, that is the relative anticanonical divisor $-K_{X/S}$ is π-ample. Then for any two $s_0,\ s_1 \in S$ the local identification $H_2(X_{s_0},\mathbb{R}) = H_2(X_{s_1},\mathbb{R})$ and $H^2(X_{s_0},\mathbb{R}) = H^2(X_{s_1},\mathbb{R})$ yields $\mathcal{C}(X_{s_0}) = \mathcal{C}(X_{s_1})$ and $\mathcal{P}(X_{s_0}) = \mathcal{P}(X_{s_1})$*

The above result is an immediate consequence of the following main theorem from [Wiś91, Theorem 1.7], see also [Wiś98] where the ideas of [Wiś91], based on playing Hard Lefschetz Theorem against Mori's theory of rational curves, see [Mor79], [Mor82], were further developed. The notation in the present paper is consistent with that of [Wiś91]. Recall that given an ample line bundle $\mathcal{L}$ on a manifold X its nef value, or nef threshold, $\tau(\mathcal{L})$ is the infimum of $t \in \mathbb{Q}$ such that $K_X + t\mathcal{L}$ is ample $\mathbb{Q}$-divisor. Note that $\tau(\mathcal{L})$ is positive if and only if K_X is not nef.

Theorem 2. *Let us assume that $\pi : X \to S$ is a smooth (connected) family of projective manifolds with $\mathcal{L}$ a π-ample line bundle. If $\tau(\mathcal{L}_{s_0})$ is positive for some $s_0 \in S$ then the function $S \ni s \mapsto \tau(\mathcal{L}_s)$ is constant.*

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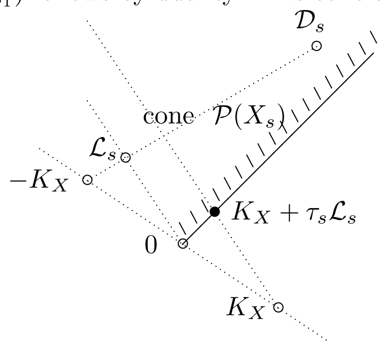
In view of Theorem 2 a proof of Theorem 1 can be reduced to choosing appropriate π-ample line bundle $\mathcal{L}$. Namely, suppose that D_0 is an ample divisor on X_{s_0}. Now, possibly shrinking S to a connected base over which the family π is topologically trivial we can extend D_0 to a divisor $\mathcal{D}$ over X; this extension provides us with an identification $H^2(X_{s_0},\mathbb{R}) = H^2(X_{s_1},\mathbb{R})$. We claim that for a sufficiently small positive ϵ the divisor $\mathcal{L} = -K_X + \epsilon \cdot (\mathcal{D} + K_X)$ is π-ample. Indeed, since ampleness is an open condition, for any $s \in S$ there exists $\epsilon_s > 0$ such that $-K_X + \epsilon_s \cdot (\mathcal{D} + K_X)$ is π-ample in an open Zariski neighborhood of X_s. Thus we can take $\epsilon = \min(\epsilon_{s_0}, \epsilon_{s_1})$ and possibly shrink S to a smaller connected variety containing both s_0 and s_1. We denote the resulting restricted family as before: $\pi : X \to S$. Now we apply Theorem 2 to argue that $\mathcal{D}_{s_1}$ is ample: see the picture below which presents a plane in $N^1(X_s)$ spanned on K_X and $\mathcal{D}$; it is apparent that invariance of τ_s yields ampleness of $\mathcal{D}_s$ because the line determined by $K_X + \tau_s L_s$ is fixed and it constitues a border of the nef cone for all s. This implies that, in terms of the identification $H^2(X_{s_0},\mathbb{R}) = H^2(X_{s_1},\mathbb{R})$, we get $\mathcal{P}(X_{s_0}) = \mathcal{P}(X_{s_1})$. The equality $\mathcal{C}(X_{s_0}) = \mathcal{C}(X_{s_1})$ follows by duality. This concludes a proof of Theorem 1.



I would like to thank Rob Lazarsfeld who convinced me that it is worthwhile to write this note. He also observed that rigidity of the cone of big divisors for Fano manifolds follows from an extension theorem by Siu, [Siu02], and suggested the following corollary to Theorem 2.

Theorem 3. *Let $\pi : X \to S$ be a smooth family of projective manifolds. If for some $s_0 \in S$ the variety X_{s_0} is not minimal, that is K_{s_0} is not nef, then no variety X_s is minimal.*

A version of the above theorem was proved in analytic category in [AP97].

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