On deformation of nef values

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Introduction.

Let L be an ample line bundle over a smooth projective variety X. It follows from a theorem of Kawamata that if the canonical divisor K_X of X is not nef then there exists a positive rational number τ, which we call the *nef value* of L, such that the $\mathbf{Q}$-divisor $K_X+\tau L$ is nef but not ample. From the Contraction Theorem of Kawamata and Shokurov, it folows that $K_X+\tau L$ is actually semi-ample. The map associated to the linear system $|m(K_X+\tau L)|$, for $m \gg 0$, is usually called the *adjunction morphism* and is the object of an extensive study, see for example [BFS] for an overview of the present state of the adjunction theory.

In the present paper we prove that the *nef value* is invariant under deformation, Theorem (1.7). This result has its roots in the theory of extremal rays on smooth manifolds, see Proposition (1.3), and is obtained by playing the Strong Lefschetz Theorem, (1.1), against an estimate on the locus of an extremal ray, (1.2). Examples (1.4) and (2.3) show the boundary cases for which the theory fails. As an application, in Section 2, we make some improvement upon a result of Ein, Theorem (2.4). This is a partial answer to a conjecture stated by Beltrametti and Sommese in connection with the characterization of adjunction morphisms, see (2.6) and (2.9).

The present paper was conceived when I was preparing a talk on applications of the Strong Lefschetz Theorem in the proofs of Barth-Lefschetz type results according to papers of Hartshorne, Sommese and Lazarsfeld. I was very much influenced by these papers. The talk was prepared for the Autumn School of Algebraic Geometry which took place at Rajgród, Poland. I would like to thank the organizers of the conference for providing nice working conditions and stimulating atmosphere.

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Section 1.

(1.0). The set up: In the present paper all varieties are defined over the field of complex numbers. We consider a family of smooth projective varieties $\pi : X \to S$ parameterized by a smooth irreducible variety S; that is, X is a smooth variety, projective over S and the map π is smooth so that the fibers of π are projective manifolds. We fix a closed point $0 \in S$. The variety S does not have to be complete. For example, S can be the spectrum of a discreet valuation ring or, equivalently, a germ of a smooth curve. By X_0 we will denote the fiber over the special point 0 while by X_s we will denote a general fiber of the map π.

For such a family we use the language and the notation from [KMM]. For example, in the space $N_1(X/S)$ of relative 1-cycles we consider the cone $NE(X/S)$ of effective 1-cycles relative to S. Similarly we use the notion of π-nef , π-semi-ample and π-ample line bundles. For example, a multiple of a π-semi-ample line bundle L is π-generated, that is, we have an epimorphism

$$\pi^*\pi_* L^{\otimes m} \to L^{\otimes m} \to 0$$

for $m \gg 0$ and therefore it defines a projective S-morphism. In other words, to any π-semi-ample line bundle L we associate a surjective morphism of S-schemes $\varphi : X \to Y$ and, taking the above m large enough, we may assume that Y is normal and φ has connected fibers. Following [S] we consider also the notion of π-k-ample line bundle, that is, a line bundle L which is π-semi-ample and such that the associated projective S - morphism has all fibers of dimension $\leq k$. Let us note that the restriction L_s of a π-semi-ample, respectively π -ample or π-k-ample to any fiber X_s over a closed point $s \in S$ remains semi-ample, respectively ample or k-ample. Finally, let us recall that a 0-ample line bundle is just an ample line bundle.

If no confusion is likely we will use the notion of a divisor and a line bundle on a smooth variety interchangebly. We will also use the notion of a **Q**-divisor i.e. a divisor with **Q**-coefficients whose multiple is a Cartier divisor. By a map associated to a semi-amle **Q**-divisor we will understand a map defined by the complete linear system of some large multiplicity of such divisor.

(1.1) Key Lemma. *Let $\pi : X \to S$ be as above. Assume that L is a π-semi-ample line bundle on X such that its restriction L_s is k-ample on a general fiber X_s. Let $\varphi : X \to Y$ be the morphism associated to L. Then for any irreducible subvariety Z of X_0 we have*

$$2 \cdot dim Z - dim \ \varphi(Z) \leq dim X_0 + k.$$

Proof. The proof depends on a version of the Strong Lefschetz Theorem as discussed in [S]. Let us denote $dim X_0 = n$, $dim Z = a$ and $dim \varphi(Z) = b-1$. Let $\eta_Z \in H^{2n-2a}(X_0, \mathbf{C})$ be the cohomology class of the variety Z. Then for a general choice of $H_1, H_2, \ldots H_b$ from the linear system $|mL_0|, m \gg 0$ we have

$$H_1 \cap H_2 \cap \ldots \cap H_b \cap Z = \emptyset$$

and therefore

$$\eta_Z \cup (c_1 L)^{\cup b} = 0 \qquad \text{in} \qquad H^{2n-2a+2b}(X_0, \mathbf{C}).$$

On the othe hand, by the theory of deformation of smooth varieties (see e.g. [K] Theorem 2.3) the pair (X_0, L_0) is topologically the same as the pair (X_s, L_s), for a general closed point $s \in S$, that is, X_0 is diffeomorphic to X_s and $c_1 L_0$ coincides with $c_1 L_s$ in the identified groups $H^2(X_0, \mathbf{Z}) = H^2(X_s, \mathbf{Z})$. Therefore the Strong Lefschetz Theorem (see [S], Proposition 1.17) applied to L_s implies that the cup-product map

$$H^q(X_0, \mathbf{C}) \overset{\cup(c_1 L_0)^{\cup r}}{\longrightarrow} H^{q+2r}(X_0, \mathbf{C})$$

is injective for $r \le n - q - k$. Comparing this with the vanishing of $\eta_Z \cup (c_1 L)^{\cup b}$ yields the desired ineqality.

(1.2) Lemma. *Let $\pi : X \to S$ be as in (1.0). Assume moreover that R is an extremal ray of $NE(X/S)$ and let $\varphi_R : X \to Y$ be the contraction of R. If Z is an irreducible component of the locus of curves whose classes are in R then*

$$2 \cdot dimZ - dim\varphi_R(Z) \ge dimX + \ell(R) - 1$$

where $\ell(R)$ is the length of the ray R as defined in [W].

Proof. Note that the left-hand-side term of the above inequality is equal to

$$dimZ + dim(\text{general fiber of } \varphi_{R|Z}).$$

Thus the lemma is just a version of (1.1) from [W] and its proof is the same.

We summarize the above lemmata to the following

(1.3) Proposition. *In the situation of (1.0) let R be an extremal ray of $NE(X/S)$. Then the locus of curves from R dominates S. In particular, either the contraction of R is of fiber type or its exceptional locus maps via π onto S.*

Proof. We can assume that $dimS = 1$ and the locus of curves from R has non-empty intersection with the fiber X_0, $0 \in S$. Let H be a good supporting divisor of R and let $\varphi_R : X \to Y$ be the contraction of the ray R associated to H. Assume that the restriction φ_R to a general fiber X_s is finite-to-one so that H_s is ample on X_s. Then let $Z_0 \subseteq X_0$ be an irreducible component of the exceptional set of φ . From (1.2) it follows that

$$2 \cdot dimZ_0 - dim\ \varphi_R(Z_0) \ge dim\ X = dim\ X_0 + 1$$

while from (1.1) we have

$$2 \cdot dimZ_0 - dim\ \varphi_R(Z_0) \le dim\ X_0.$$

(1.4) Example. Let us note the above proposition is true only for contractions in the sense of Mori theory. To see this let us consider a deformation of Hirzebruch surfaces. Assume that S is a spectrum of a discreet valuation ring with the closed point 0 and the general point s. Over the product $\mathbf{P}^1 \times S$ we take a rank-2 vector bundle E whose restriction to $\mathbf{P}^1_0$ is $O(2a) \oplus O$ and the restriction to $\mathbf{P}^1_s$ is $O(a) \oplus O(a)$, where a is a positive integer. Let X be the projectivization of E. Then the relative hyperplane-section bundle $O(1)$ on X is ample on X_s but is not on X_0. The associated morphism contracts a $(-2a)$-curve in X_0.

(1.5). Assume that $\pi : X \to S$ is as in (1.0) and L is a π-ample line bundle. For such L we introduce the *nef-value function* on S. Namely, for any closed point $s \in S$ we associate the real number

$$\tau(s) := \inf\{t \in \mathbf{Q} : \text{the } \mathbf{Q}\text{-divisor } (K_X + tL) \text{ is ample on } X_s \}.$$

Let us note that by Kawamata's Rationality Theorem ([KMM] 4-1-1) the value of τ is rational whenever is positive. Moreover we have

(1.6) Lemma. *The function $s \mapsto \tau(s)$ is upper-semicontinous on S.*

Proof. We have to show that for any $u \in \mathbf{R}$ the set $\{s \in S : \tau(s) < u\}$ is open in S. But ampleness is an open property so the lemma follows by the equality

$$\{s \in S : \tau(s) < u\} = \bigcup_{\substack{t<u \\ t\in\mathbf{Q}}} \{s \in S : (K_X + tL)_{|X_s} \text{ is an ample } \mathbf{Q}\text{-divisor on } X_s\}.$$

The main result of this section is the following

(1.7) Theorem. *In the situation of (1.0) if the function $s \mapsto \tau(s)$ assumes a positive value on S then it is constant.*

Proof. In view of (1.6) we are only to prove that $\tau(0) \leq \tau(s)$ for a general s in S if only $\tau(0)$ is positive. As in the proof of (1.3) we may assume that $dim S = 1$ and $\tau(s) \leq \tau(0) > 0$ for any $s \in S$. Then by the Rationality and Base-Point-Free Theorems ([KMM] 3-1-1) the line bundle $L = O(m(K_X + \tau(0)))$ is a π-generated for $m \gg 0$ such that $m\tau(0) \in \mathbf{Z}$. If $\tau(0) > \tau(s)$ then the restriction L_s is ample on X_s. This contradicts (1.3). Namely, the map associated to $|mL|$, $m \gg 0$, is a contraction of an extremal face in the sense of [KMM]. Thus, if we take an extremal ray contracted by this map its locus would be contained in X_0, which by (1.3) is not the case.

Section 2.

(2.0). The Theorem (1.7) can be applied to the problem of specialization of vector bundles. For this purpose we consider a rank-r vector bundle E on the product $Y \times S$, where Y is a smooth projective variety and S is the spectrum of a discreet valuation ring, or, equivalently, a germ of a smooth curve. The closed point of S will be denoted by 0 and the general point by s. In this situation the restriction E_0 of the bundle E to the fiber Y_0 is called the specialization of the bundle $E_s := E_{|Y_s}$.

Let $X := \mathbf{P}(E)$ be the projectivization of E with the induced morphism $p : X \to Y$ and the relative hyperplane-section divisor (class) ξ_E, $O(\xi_E) = O_{\mathbf{P}(E)}(1)$. The variety X maps on making X a family of smooth varieties parameterized by S.

Let us recall that a vector bundle F over a variety Y is called nef, respectively semi-ample, ample or k-ample if the relative hyperplane-section line bundle $O(\xi_F) = O_{\mathbf{P}(F)}(1)$ on the projectivization $\mathbf{P}(F)$ is such.

(2.1) Proposition. *Assume that E_0 is a nef vector bundle over Y such that the divisor $-c_1E_0 - K_Y$ is ample on Y. If E_0 is a specialization of an ample vector bundle then E_0 is ample.*

Proof. First let us recall that the canonical divisor of the projectivisation $\mathbf{P}(E_0)$ is equivalent to $-rankE_0 \cdot \xi_{E_0} + p^*(K_Y + c_1E_0)$. In our case, since $-c_1E_0 - K_Y$ is ample and ξ_{E_0} is nef (and p-ample), it follows that $-K_{\mathbf{P}(E_0)}$ is ample. Therefore, in the situation of (2.0), the line bundle $L := O(\xi_E) \otimes O(-K_X)$ on X is π-ample and its nef value at 0 is positive. It is not hard to see that $\tau(0) \leq 1$, with equality possible only if E_0 is not ample. On the other hand, since E_s is ample it follows that $\tau(s) < 1$, thus , in view of (1.7) the Proposition follows.

(2.2) Corollary. *Let E_0 be a specialization of a rank-r trivial vector bundle on $\mathbf{P}^m$. If $E_0(1)$ is nef and $r \leq m$ then E_0 is trivial.*

Proof. In view of (2.1) the bundle $E_0(1)$ is ample. Its restriction to any line is then $O(1)^{\oplus r}$ so by 3.2.1 from [OSS] the bundle E_0 is trivial.

(2.3) Example. Note that (2.2) is not true for $r = m + 1$ since one can take the Euler sequence on $\mathbf{P}^m$

$$0 \to O(-1) \to O^{\oplus(m+1)} \to \mathbf{TP}^m(-1) \to 0,$$

or its dual, to get a non-trivial specialization of $O^{\oplus(m+1)}$. On the other hand, in [B] we have examples of a non-trivial rank-2 vector bundles on $\mathbf{P}^2$ which are specializations of a trivial bundle. It is not hard to see that any such a vector bundle E_0 with the generic splitting type $O(1) \oplus O(-1)$ has its second twist $E_0(2)$ spanned.

The above Corollary is an improvement of Ein's 1.6 from [E]. This allows to improve his Theorem 1.7 to the following

(2.4) Theorem. *Let X be a smooth subvariety of $\mathbf{P}^N$ of dimension n. Assume that there is a m-plane Π_0 in X whose normal bundle is trivial. If $m \geq n/2$ then X is has the structure of a rank-m projective bundle over a smooth projective variety of dimension $n-m$ and Π_0 is a fiber of this projective bundle.*

We will show that the above theorem is equivalent to a result on the adjunction mapping. First let us recall some definitions from the adjunction theory.

(2.5). Let X be a smooth projective variety with K_X not nef and let L be an ample line bundle on X. Let τ denote the nef value of L and assume that $\phi : X \to Y$ is the map onto a normal projective variety Y which is associated to the $\mathbf{Q}$-divisor $K_X + \tau L$, the fibers of ϕ are connected. Such a map is called *adjunction map* associated (*adjoint*) to L. In the language of Minimal Model (or Mori) Theory the map ϕ is a contraction of an extremal face. In particular L can be chosen so that ϕ is a contraction of an extremal ray (Mori contraction).

If $dimY < dimX$ and $\tau = dimX - dimY + 1$ then following [BS] we call the map ϕ a *scroll* or *adjunction scroll*. By adjunction and Kobayashi-Ochiai criterion the general fiber of the *scroll* is isomorphic to $\mathbf{P}^{dimX-dimY}$ and the restriction of L to the fiber is then isomorphic to $O(1)$. Equivalently, given any morphism $\phi : X \to Y$ with connected fibers onto a normal projective variety Y, $dimY < dimX$, ϕ is a *scroll* if

$$K_X + (dimX - dimY + 1)L = \phi^* L_Y$$

for some ample line bundle L_Y on Y. Actually, from the point of geometry of X one does not have to assume L_Y to be ample. Indeed, if L_Y is not ample we can choose an ample line bundle Λ on Y and twist the original L_Y so that $L'_Y := L_Y \otimes \Lambda^{\otimes(dimX-dimY+1)}$ is ample on Y and then taking $L' := L \otimes \phi^*\Lambda$ we obtain the *scroll* as defined above.

If $\phi : X \to Y$ is a Mori contraction then it is a scroll if its general fiber is isomorphic to $\mathbf{P}^{dimX-dimY}$ and the restriction of L to the fiber is isomorphic to $O(1)$. Indeed, in this case the restriction of $K_X + (dimX - dimY + 1)L$ to the fiber is trivial so it must be a pull-back of a line bundle from Y.

Using the notion of *scroll* we reformulate (2.4) to the following

(2.6) Theorem. *Let X be a smooth projective variety of dimension n and L a very ample line bundle on X. Assume that the map $\phi : X \to Y$ adjoint to L is a scroll. If $dimY \leq n/2$ then Y is smooth and ϕ is a projective bundle over Y.*

Let us note that the theorems (2.4) and (2.6) are equivalent. Indeed, the implication (2.4) $\Rightarrow$ (2.6) is clear, while the implication (2.6) $\Rightarrow$ (2.4) follows from the following result from [BSW]:

(2.7) Theorem. *([BSW] 2.5 and 3.2.1). Let X be a smooth projective variety of dimension n and let L be an ample line bundle over X whose nef value we denote by τ. Assume that X contains a smooth subvariety Π_0 isomorphic to $\mathbf{P}^m$ such that the restriction $L_{|\Pi_0}$ is isomorphic to $O(1)$. Assume moreover that the normal bundle of Π_0 is trivial.*

If $m \geq (n-1)/2$ then either $m = (n-1)/2$, $\tau = n-m+1 = m+2$ and $X \cong \mathbf{P}^m \times \mathbf{P}^{n-m}$ or $\tau = m+1$ and one of the following is true:

(2.7.1) the map adjoint to L, $\phi : X \to Y$, is a scroll and extremal ray (Mori) contraction with Π_0 being one of its fibers,

(2.7.2) one of the cases discussed in (2.5.2) and (2.5.3) of [BSW] occurs and X has a projective bundle structure (not adjoint to L) such that Π_0 is a fiber of this projective bundle.

Proof. First we claim that X is covered by lines. If ℓ is a line contained in Π_0, then the normal bundle of ℓ is isomorphic to $O(1)^{\oplus(m-1)} \oplus O^{\oplus(n-m)}$ so that it is generated at every point and its first cohomology group vanishes. Therefore, by the theory of deformation of cycles, curves obtained by deforming ℓ cover a neighbourhood of Π_0. Moreover, as $\ell.L = 1$, it follows that any limit of these curves is again an irreducible rational curve so that they form a non-breaking family in the sense of [BSW]. Note that

$$\nu := -K_X.\ell - 2 = m - 1 \geq (n-3)/2$$

so that we are in the situation of Theorem (2.5) from [BSW] and the result follows. The statement in (2.7.2) concerning the projective bundle structure follows easily from the explicit description of the cases quoted here as (2.5.2) and (2.5.3), see also (2.6) in [BSW] as well as [W1].

For the proof of (2.6) we will need the following

(2.8) Lemma. *Let X be a smooth projective variety of dimension n and let L be an ample line bundle over X. Assume that the map $\phi : X \to Y$ adjoint to L is an adjunction scroll and also a contraction of an extremal ray. Let $\Pi_0 \cong \mathbf{P}^m$ be a general fiber of ϕ, m denoting $dimX - dimY$. Assume that every m-cycle Π_t from the component T of $Hilb_X$ containing Π_0 satisfies the following condition*

> *(*) Π_t is a smooth subvariety of X, $\Pi_t \cong \mathbf{P}^m$, $L_{|\Pi_t} \cong O(1)$ and the normal bundle to Π_t is trivial,*

then Y is smooth and $\phi : X \to Y$ is a $\mathbf{P}^m$-bundle.

Proof. We follow Ein's arguments from the proof of 1.7 from [E]. First let us note,that from the properties of $Hilb_X$ it follows that T is smooth and of dimension $n-m$. Over T we have a universal family Π of m-cycles whose projection onto T makes Π a $\mathbf{P}^m$-bundle over T. The variety Π admits also a natural map $h : \Pi \to X$.

We claim that the map h is birational. Indeed, over a non-empty open subset U of Y the map ϕ is flat so by the universal property of Hilbert scheme we have a map $U \to T$ which by base-change gives the inverse of h on $\phi^{-1}(U)$.

On the other hand, using the natural properties of $Hilb_X$, as in [E], p. 901, we conclude that the derivative of the map h is an isomorphism at every point. Therefore h is actually an isomorphism.

Finally, since $\rho(X) = \rho(\Pi) = \rho(T) + 1$, ρ denoting the Picard number, it follows that the map $X \cong \Pi \to T$ is a Mori contraction of X so that it must concide with ϕ. In particular $Y \cong T$ is smooth and ϕ is a $\mathbf{P}^m$-bundle.

Proof of (2.6). According to (2.7) the map $\phi : X \to Y$ is a contraction of an extremal ray of X. Thus, in view of (2.8), we will be done if we show that the condition (*) of (2.8) is satisfied for any m-cycle Π_t obtained by deforming a general fiber Π_0 of ϕ. A deformation of a linear subspace in $\mathbf{P}^N$ remains linear, so we have only to check that the normal bundle of $\Pi_t \cong \mathbf{P}^m$ is trivial. This is clearly true for a general choice of t. Thus for any m-cycle Π_t its normal bundle is a specialization of the trivial one.

On the other hand, from the sequence of embeddings $\Pi_t \hookrightarrow X \hookrightarrow \mathbf{P}^N$, the latter defined by L, we have the following sequence of conormal bundles

$$0 \to (N^*_{X/\mathbf{P}^N})_{|\Pi_t} \to N^*_{\Pi_t/\mathbf{P}^N} \cong O_{\Pi_t}(-1)^{\oplus(N-m)} \to N^*_{\Pi_t/X} \to 0.$$

Therefore $N^*_{\Pi_t/X}(1)$ is spanned on $\Pi_t \cong \mathbf{P}^m$ and thus it satisfies assumptions of (2.2). Hence $N_{\Pi_t/X}$ is trivial and we are done.

(2.9) Remark. Beltrametti and Sommese [BS] conjectured that

> in the situation of (2.5) if $\phi : X \to Y$ is a scroll and $dimX \geq 2dimY + 1$ then Y is smooth and $\phi : X \to Y$ is a projective bundle over Y.

The low–dimensional cases of the conjecture are done, see [BS] or [BSW]. From (2.7) it follows that for $dimX \geq 2dimY + 1$ the scroll $\phi : X \to Y$ is a Mori contraction; therefore, to prove this conjecture, one should check (*) of (2.8).

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