Pragmatic 2010 notes, examples, questions
Part II: structure of cones

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based on a joint project with Klaus Altmann

conology: Mori view

Let Z be a $\mathbb{Q}$-factorial projective variety over $\mathbb{C}$ such that $\mathrm{Cl}(Z)$ is a lattice. We define vector space $N^1(Z) = \mathrm{Cl}(Z) \otimes \mathbb{R} = \mathrm{Pic}(Z) \otimes \mathbb{R}$ and inside this space we have closed cones:

- $\mathrm{Eff}(Z)$ spanned by classes of effective divisors
- $\mathrm{Mov}(Z)$ movable divisors, with no base components
- $\mathrm{Nef}(Z)$ nef divisors, closure of ample cone

The dual $N_1(Z)$ is the space of 1-cycles and dual of $\mathrm{Nef}(Z)$ is the cone of effective 1-cycles (Kleiman).

Fano varieties

If $-K_Z \in \operatorname{int} \operatorname{Nef}(Z)$ then Z is a Fano variety. Then (assuming good singularities) $\operatorname{Nef}(Z)$ is a rational polyhedral cone and there is a bijection

- contractions $\varphi : Z \to Y_\varphi$ surjective morphisms with connected fibers onto normal varieties
- faces of $\operatorname{Nef}(Z)$ defined by intersection $\operatorname{Nef}(Z) \cap \varphi^* N^1(Y_\varphi) = \varphi^* \operatorname{Nef}(Y_\varphi)$.

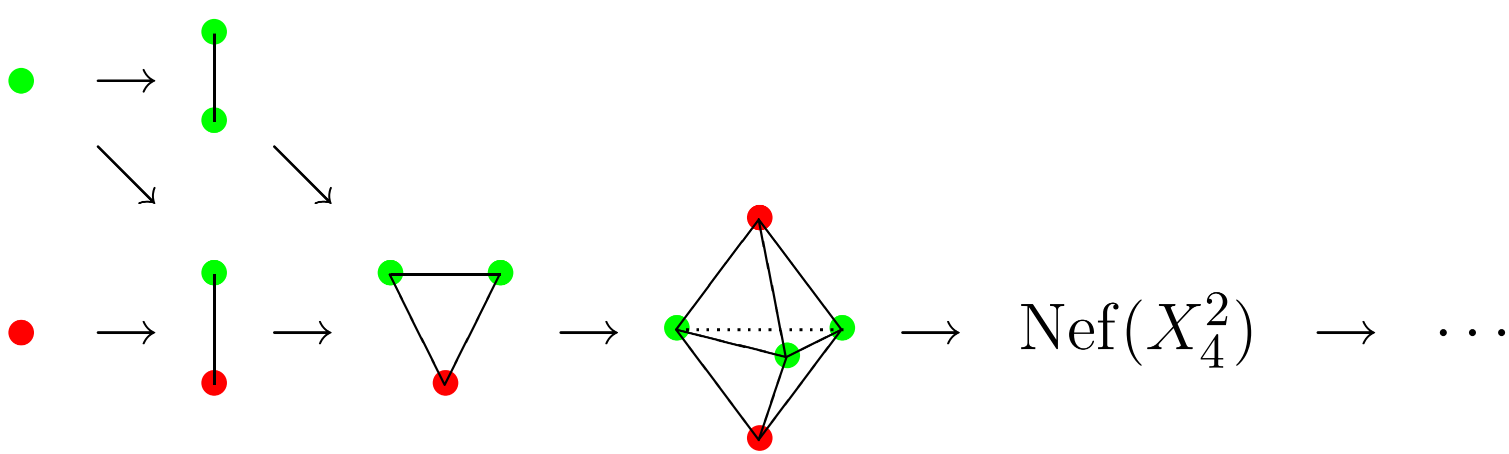
Instead of polyhedral cones we will consider their compact sections, polytopes.

1st example: Del Pezzo surf

Let X_r^2 be $\mathbb{P}^2$ blow-up at r points. We have contractions

$$\begin{array}{ccccccccccc}
\mathbb{P}^1 & \leftarrow & \mathbb{P}^1\times\mathbb{P}^1 & & & & & & & & \\
 & \nwarrow & & \nwarrow & & & & & & & \\
\mathbb{P}^2 & \leftarrow & X_1^2 & \leftarrow & X_2^2 & \leftarrow & X_3^2 & \leftarrow & X_4^2 & \leftarrow & \cdots
\end{array}$$

And a sequence of polytopes



marked polytopes

- We mark faces of a polytope Δ of dimension r by non-negative integers. The interior of the polytope is marked by 0 and if β is a face of α then marking of β is not smaller than that of α.
- Marked face polynomial $P_\Delta(x,y) = \sum a_{ij}x^iy^j$ where a_{ij} is the number of faces of codim i marked by j.
- Let Δ_r^2 be a section of $\mathrm{Nef}(X_r^2)$. A face associated to $\varphi : X_r^2 \longrightarrow Y_\varphi$ is marked by $\dim X_r^2 - \dim Y_\varphi$.
- Let P_r^2 be the marked polynomial of Δ_r^2. We see $P_2^2 = 1 + 3x + x^2 + 2x^2y$ and $P_3^2 = 1 + 6x + 9x^2 + 2x^3 + 3x^3y$.

recursive relations

Theorem. [Stalij, MSc thesis 2001]
For $r = 2, \ldots, 8$ the following holds

- The polytope Δ_r^2 is simple (dual to a simplex) at faces marked by 0 while dual to a cross-polytope (orthoplex or co-cube) at vertices marked by 1.
- The polynomials P_r^2 satisfy equations
  $\partial_x P_r^2(x,0) = \partial_x P_r^2(0,0) \cdot P_{r-1}^2(x,0)$ and
  $2(r-1) \cdot \partial_y P_r^2(1,0) = \partial_x P_r^2(0,0) \cdot \partial_y P_r^2(1,0)$
- If $P_r^2(-1,1) = (-1)^{r+1}$ and $P_{r-1}^2(x,0) < P_r^2(x,0)$ then the equations have unique solution for $r < 8$ and no solution for $r = 9$.

geometric contents

The theorem follows from easy

Proposition.

- For $r \geq 2$ every contraction of X_r^2 factors through a blow-down of (-1)-curve, $X_r^2 \to X_{r-1}^2$
- Contractions to $\mathbb{P}^2$ contract r disjoint (-1)-curves.
- Contractions to $\mathbb{P}^1$ have $r-1$ reducible fibers (conics) consisting of two (-1)-curves.
- The number of (-1) curves on X_r^2 is $\partial_x P_r^2(0,0)$

Gosset polytopes (1900)

The dual of Δ's are known: Gosset (1900), Coxeter (1940): k_{21} family of semi-regular polytopes, see

http://en.wikipedia.org/wiki/Semiregular_E-polytope

Gosset polytopes (1900)

Elements

Gosset semiregular figures

n-ic	k_{21}	Graph	Name Coxeter-Dynkin diagram	Facets: (n-1)-simplex $\{3^{n-2}\}$	Facets: (n-1)-orthoplex $\{3^{n-4,1,1}\}$	Elements: Vertices	Edges	Faces	Cells	4-faces	5-faces	6-faces	7-faces
3-ic	-1_{21}		Triangular prism	2 triangles	3 squares	6	9	5					
4-ic	0_{21}		Rectified 5-cell	5 tetrahedron	5 octahedron	10	30	30	10				
5-ic	1_{21}		Demipenteract	16 5-cell	10 16-cell	16	80	160	120	26			
6-ic	2_{21}		2_{21} polytope	72 5-simplexes	27 5-orthoplexes	27	216	720	1080	648	99		
7-ic	3_{21}		3_{21} polytope	576 6-simplexes	126 6-orthoplexes	56	756	4032	10080	12096	6048	702	
8-ic	4_{21}		4_{21} polytope	17280 7-simplexes	2160 7-orthoplexes	240	6720	60480	241920	483840	483840	207360	19440
9-ic	5_{21}		E8 lattice	∞ 8-simplexes	∞ 8-orthoplexes	∞	∞	∞	∞	∞	∞	∞	∞

Cox rings and MDS

- Let Z be a $\mathbb{Q}$-factorial projective variety over $\mathbb{C}$ such that $\mathrm{Cl}(Z)$ is a lattice. We define

$$\mathrm{Cox}(Z) = \bigoplus_{D \in \mathrm{Cl}(Z)} \Gamma(Z, \mathcal{O}(D))$$

  with multiplicative structure defined by a choice of divisors whose classes form a basis of $\mathrm{Cl}(Z)$.

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  with multiplicative structure defined by a choice of divisors whose classes form a basis of $\mathrm{Cl}(Z)$.
- Assume $\mathrm{Cox}(Z)$ finitely generated and call Z a Mori Dream Space (or MDS). The $\mathrm{Cl}(Z)$-grading of $\mathrm{Cox}(Z)$ yields action of torus $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(Z), \mathbb{C}^*) \cong (\mathbb{C}^*)^{\mathrm{rk}(\mathrm{Cl}(Z))}$ on the affine variety $\mathrm{Spec}(\mathrm{Cox}(Z))$.

MDS and SQM's

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MDS and SQM's

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MDS and SQM's

An MDS Z has finitely many small (iso in codim 1) $\mathbb{Q}$-factorial modifications Z_i (SQM's) [Hu, Keel]
Z_i share the same Cox ring and, by strict transform, we identify $\mathrm{Div}(Z_i)$ and $\mathrm{Cl}(Z_i)$ with $\mathrm{Div}(Z)$ and $\mathrm{Cl}(Z)$, respectively; same holds for effective and movable cones

$$\mathrm{Eff}(Z_i) = \mathrm{Eff}(Z) \qquad \mathrm{Mov}(Z_i) = \mathrm{Mov}(Z)$$

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$$\mathrm{Eff}(Z_i) = \mathrm{Eff}(Z) \qquad \mathrm{Mov}(Z_i) = \mathrm{Mov}(Z)$$

However, the cones $\mathrm{Nef}(Z_i)$ are different, that is $\mathrm{int}\,\mathrm{Nef}(Z_i) \cap \mathrm{int}\,\mathrm{Nef}(Z_j) = \emptyset$ if $Z_i \neq Z_j$ and we have decomposition

$$\mathrm{Mov}(Z) = \bigcup_i \mathrm{Nef}(Z_i)$$

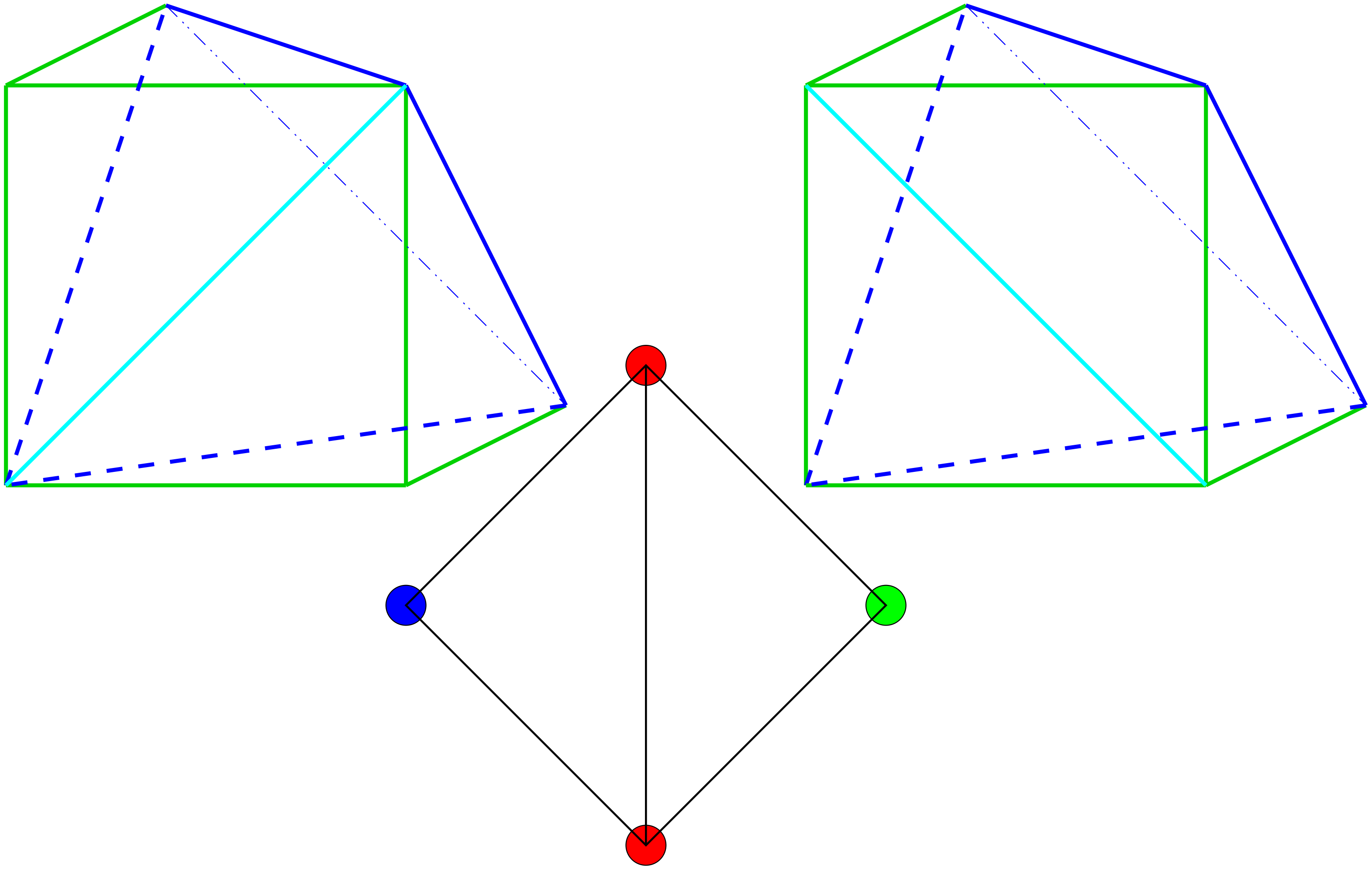
Fano spaces

If $-K_Z \in \operatorname{int}\operatorname{Mov}(Z)$ then an MDS Z is called a Fano space. The set of all its SQM's will be denoted by $\mathcal{Z}$ There is a bijection

- classes (!) of divisorial or fiber type contractions $\varphi : Z_i \to Y_{\varphi,i}$, where $Z_i \in \mathcal{Z}$ and $Y_{\varphi,i} \in \mathcal{Y}_\varphi$ (call them essential contractions)
- faces of $\operatorname{Mov}(Z)$ defined by intersection $\operatorname{Mov}(Z) \cap \varphi^* N^1(Y_\varphi) = \varphi^* \operatorname{Mov}(Y_\varphi)$.

Instead of polyhedral cones we will consider, as usually, their compact sections, polytopes. We color vertices by targets of the contractions $\bullet = \mathbb{P}^1$, $\bullet = \mathbb{P}^2$, $\bullet = \mathbb{P}^3$.

ex: bl-up $\mathbb{P}^3$ in 2 pts



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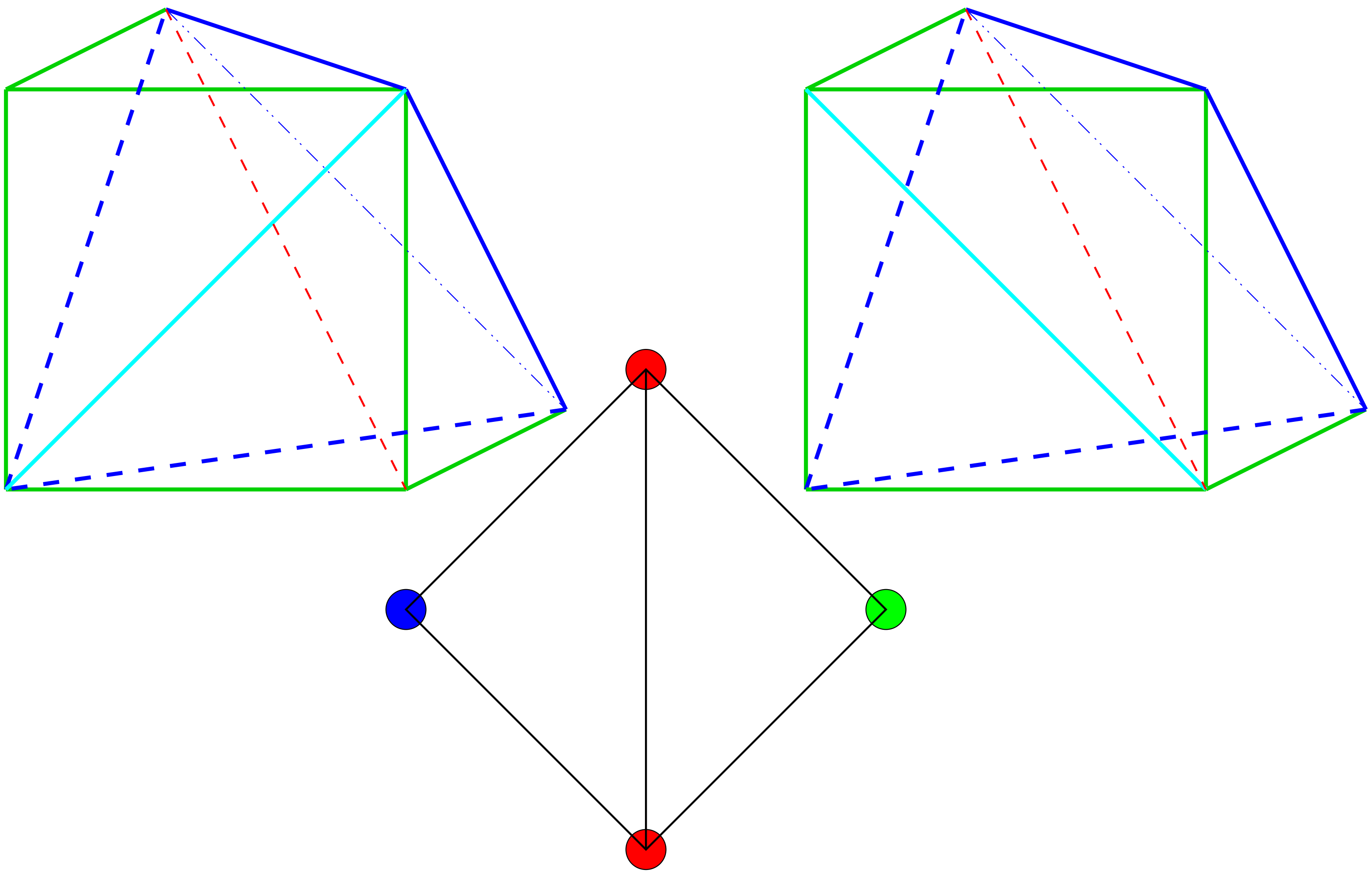

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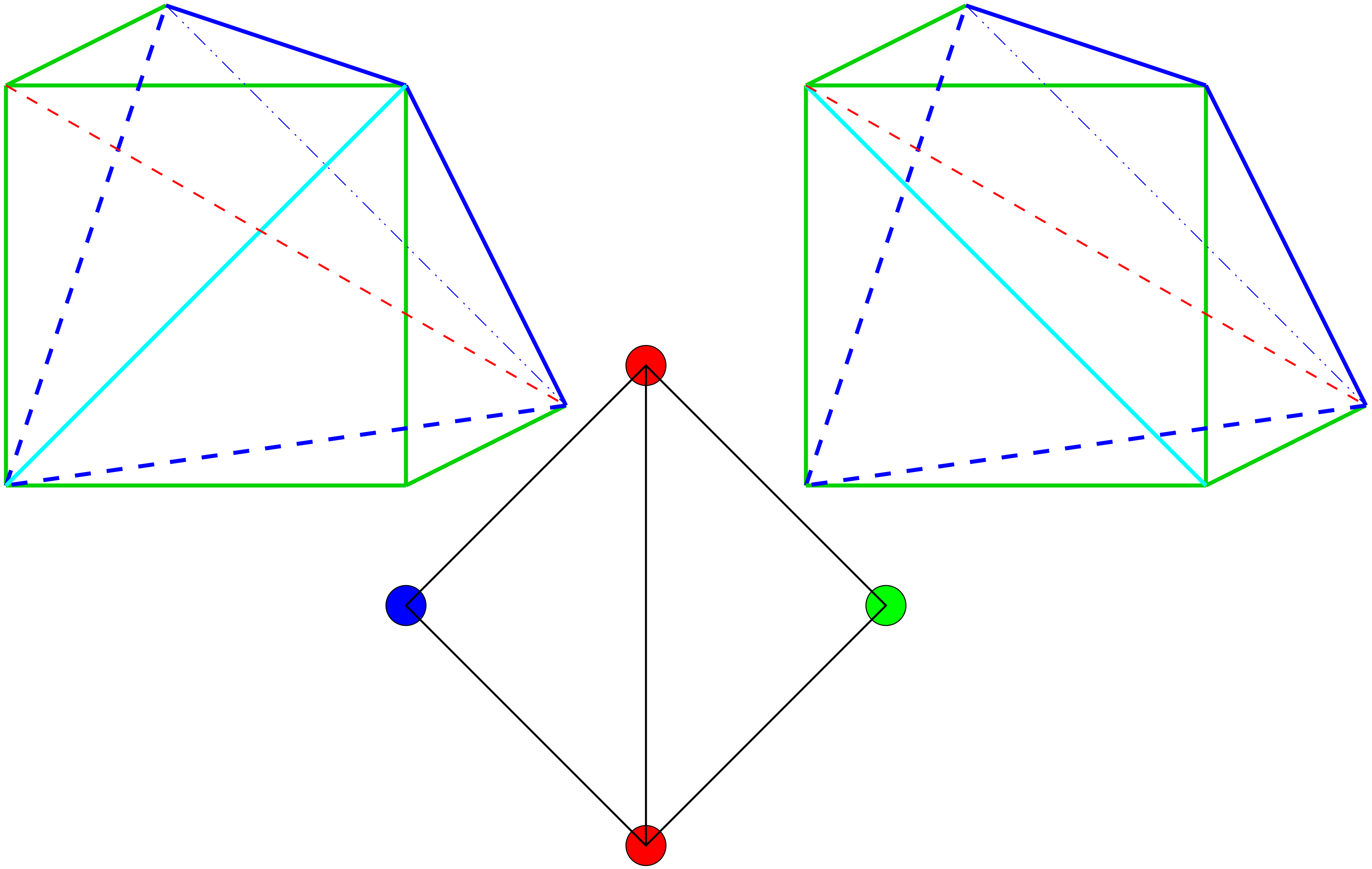

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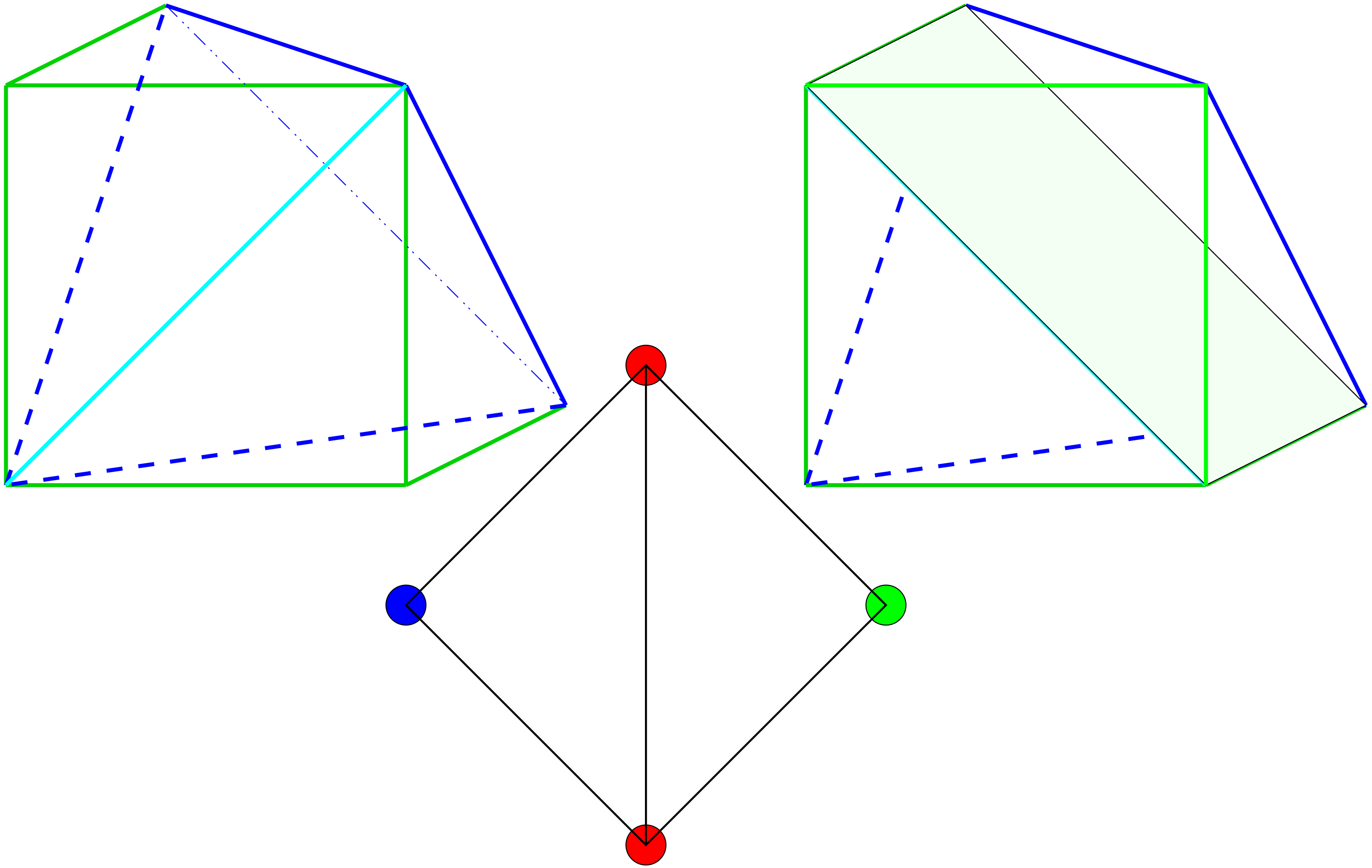


bl-up $\mathbb{P}^3$ in 2 pts: convexity

Notations: $N = \mathbb{Z}^3$ with basis e_1, e_2, e_3 and add $e_0 = -e_1 - e_2 - e_3$, they span rays of $\mathbb{P}^3$ fan. Two blow-ups add rays spanned by, say $f_1 = -e_1$ and $f_2 = -e_2$. We have the sequence

$$0 \longrightarrow \mathbb{Z}^3 = P^\vee \longrightarrow \widehat{N} = \mathbb{Z}^6 \stackrel{\pi}{\longrightarrow} N \longrightarrow 0$$

with π evaluating generators of the lattice to the generators of rays (denoted by the same letters, although, perhaps we should use $\widehat{\ \ \ }$). The kernel π is generated by $e_0 + e_1 + e_2 + e_3$, $e_1 + f_1$ and $e_2 + f_2$.

bl-up $\mathbb{P}^3$ in 2 pts: convexity

Now the convexity condition from the previous lecture are determined by the negativity of a function $u_o \in M$ on the following vectors $e_0 - f_1 + e_2 + e_3$, $e_0 + e_1 - f_2 + e_3$, $e_1 + f_1$, $e_2 + f_2$. They determine the faces of the movable cone. Note that they are relations coming from extremal curves. Look at Example 2
in www.mimuw.edu.pl/∼jarekw/java/Pragmatic2010JavaView.html

toric Mov and Ess

Take sequence

$$0 \longrightarrow P^{\vee} \longrightarrow \widehat{N} \stackrel{\pi}{\longrightarrow} N \longrightarrow 0$$

with π evaluating generators of the lattice to the generators of rays of a fan of a toric variety Z, we denote both the generators of $\widehat{M}$ and their images by e_i. Let

$$0 \longrightarrow M \longrightarrow \widehat{M} \stackrel{\psi}{\longrightarrow} P \longrightarrow 0$$

be the dual sequence. By σ^+ we denote the positive octant (in $\widehat{M}_{\mathbb{R}}$) and by $\sigma_i^+ = \sigma^+ \cap e_i^{\perp}$ we denote its facets.

toric Mov **and** Ess

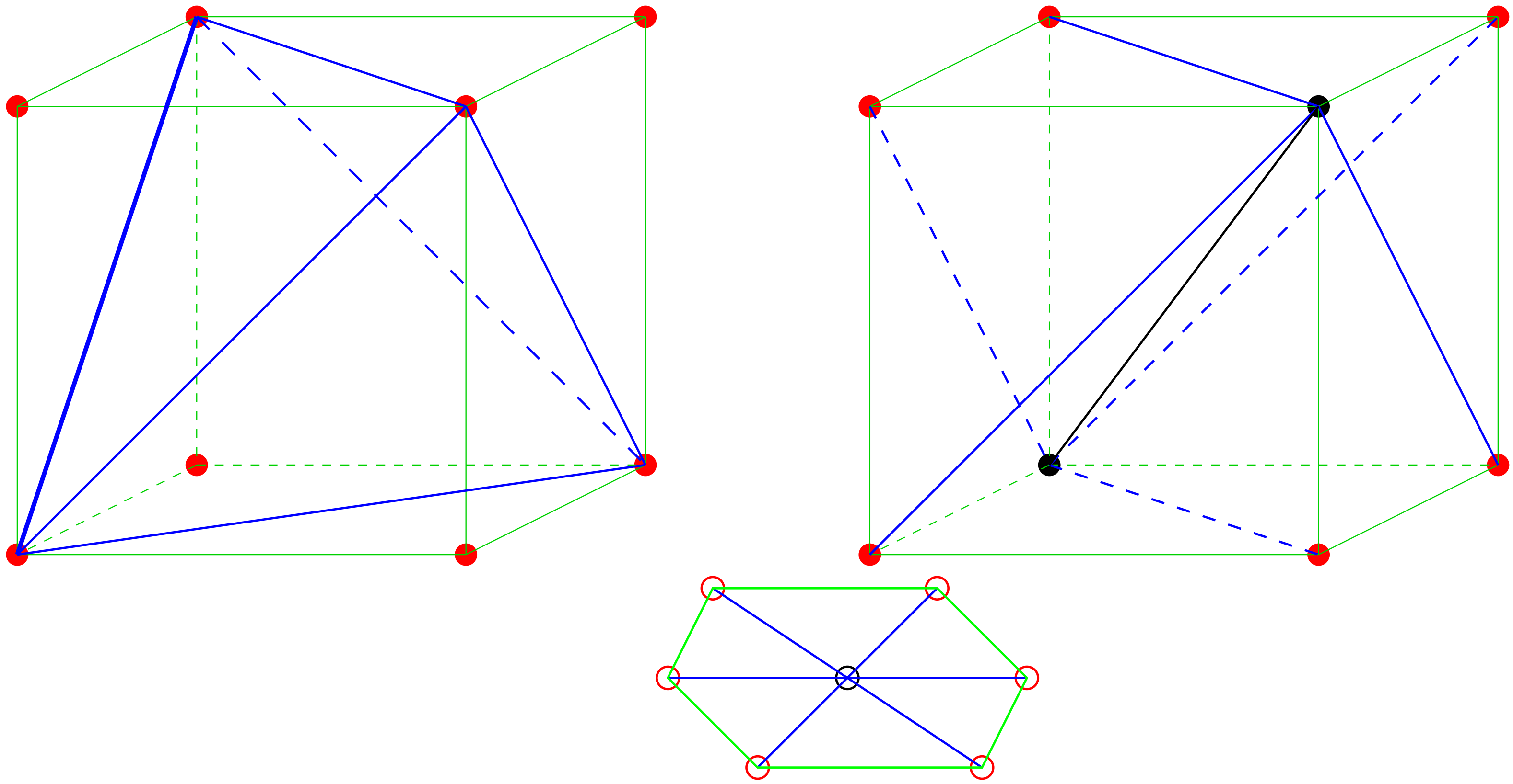
- $\operatorname{Mov}(Z) = \bigcap_i \psi(\sigma_i^+)$
- $\operatorname{Ess} = -\operatorname{Mov}^\vee$ paramatrizes relations for curves wich are effective on every SQM model Z_i
- Ess consists of vectors $\sum a_i e_i \in P_{\mathbb{R}}^\vee$ for which at most one a_i is negative.
- division of the cone Mov into Mori chambers $\operatorname{Nef}(Z_i)$ is determined by hyperplanes of the type $(\sum a_i e_i)^\perp$ for which at least two a_i's are negative/positive

bl-up $\mathbb{P}^3$ in 4 pts

Take fan in $\mathbb{Z}^3_{\mathbb{R}}$ with rays generated by e_i and $f_i = -e_i$, where $i = 0, \ldots 3$, with relation $\sum e_i = 0$.

- classes generating Ess:
  - 8 “divisorial” classes: $e_i + e_j + e_k - f_l$ and $f_i + f_j + f_k - e_l$ for quadruples
  - 4 “fiber type” classes: $e_i + f_i$
- elementary contractions of Fano spaces $\mathcal{X}^3_4$
  - 8 onto $\mathcal{X}^3_3$
  - 4 onto X^2_3
- 2×6 “flopping” classes: $\pm(e_i + e_j - f_k - f_l)$

bl-up $\mathbb{P}^3$ in 4 pts



Look at Example 3

in www.mimuw.edu.pl/∼jarekw/java/Pragmatic2010JavaView.html

Mori chambers

The cone $\mathrm{Mov}(\mathcal{Z})$ is divided into $\mathrm{Nef}(Z_i)$, the division determined by flopping (flipping) extremal curves. Dual cones are cones of effective 1-cycles $\mathrm{Eff}_1(Z_i)$ and $\mathrm{Ess}(\mathcal{Z}) = \bigcap \mathrm{Eff}_1(Z_i)$. Each $\mathrm{Eff}_1(Z_i)$ is a pointed cone - contains no non-trivial linear space.

Mori chambers

In case of $\mathcal{X}_4^3$ we have the following geometric condition: take regular octahedron and mark its vertices by $+$ and $-$, now the convex hull of $+$ vertices and $-$ vertices should not meet. Thus the SQM models of $\mathcal{X}_4^3$ are divided into the following classes

- 2 copies of X_4^3 (all vertices with the same sign)
- 2×8 copies of X_4^3 with one line flopped (all but one vertices with the same sign)
- 2×12 related to 4 vertices of one sign and 2 of the other
- 8 related to 3 vertices of one sign and 3 of the other

marked polynomials, induction

Take X^n_{n+1} which is $\mathbb{P}^n$ blown-up at $n+1$ points and the associated SQM models; the marked F polynomial of the Fano space $\mathcal{X}^n_{n+1}$ is

$$\sum_{m=0}^{n-2} \binom{n+1}{m} \cdot B_{n-m} \cdot x^m y^m \; + \; \binom{n+1}{n-1} x^{n+1} y^{n-1}$$

where B describes the "birational" part

$$B_n = 1 + 2(n+1)x + (n+1)^2 x^2 + 2 \cdot \sum_{i=3}^{n+1} \binom{n+1}{i} x^i$$

Manin-Dolgachev-Mukai table

Root systems associated to $\mathbb{P}^d$ blown-up at r generic pts.

	0	1	2	3	4	5	6	7	8
$\mathbb{P}^1$									
$\mathbb{P}^2$		toric		$\mathbb{A}_1 \times \mathbb{A}_2$	$\mathbb{A}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\mathbb{P}^3$			toric		$\mathbb{A}_1 \times \mathbb{A}_3$	$\mathbb{A}_5$	$\mathbb{D}_6$	$\mathbb{E}_7$	nfg
$\mathbb{P}^4$				toric		$\mathbb{A}_1 \times \mathbb{A}_4$	$\mathbb{A}_6$	$\mathbb{D}_7$	$\mathbb{E}_8$

- $d = 2$, $r \leq 8$: del Pezzo (1885), Manin (1972)
- $d \geq 3$, $r \leq d + 3$: Dolgachev (1978), Mukai (2003), Matsuki (1995)
- nfg $=$ not finitely generated

root systems

Manin-Mukai construction:

- Let $X = X_r^d$ be a blow up of $\mathbb{P}^d$ in r points. In the Picard lattice, or $\mathrm{Cl}(X)$, or $H^2(X, \mathbb{Z})$, we have the pull back of the hyperplane class h, the classes of exceptional divisors e_i and the anti-canonical class $-\omega_X = (d+1)h - (d-1)\sum_i e_i$.
- We define the intersection form in which these classes are orthogonal and moreover $e_i^2 = -1$ and $h^2 = d-1$. We take classes in the lattice orthogonal to $-\omega_X$ whose selfintersection is -2. They give the root system in question.

Cremona transformations

Kantor (1885-1895) studied subgroups of Cremona group of birational transformations of $\mathbb{P}^2$. They admit reflection action in the cohomology of the surface which is the resolution of the map. They are $\mathbb{A}_1 \times \mathbb{A}_2$, $\mathbb{A}_4$, $\mathbb{D}_5$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.

Coble (1917) extended this to higher dimensions. Next generalized by Dolgachev (1978) and Mukai (2001).

Mori's view

We get a diagram of essential elementary contractions of Fano spaces associated to X_r^d

$$\begin{array}{ccccccc} \mathcal{X}_{d+1}^d & \leftarrow & \mathcal{X}_{d+2}^d & \leftarrow & \mathcal{X}_{d+3}^d & & \\ & \nwarrow & & \nwarrow & & \nwarrow & \\ \cdots & \leftarrow & \mathcal{X}_{d+2}^{d+1} & \leftarrow & \mathcal{X}_{d+3}^{d+1} & \leftarrow & \mathcal{X}_{d+4}^{d+1} \end{array}$$

and a diagram of inclusion of facets

$$\begin{array}{ccccccc} \mathrm{Mov}(\mathcal{X}_{d+1}^d) & \rightarrow & \mathrm{Mov}(\mathcal{X}_{d+2}^d) & \rightarrow & \mathrm{Mov}(\mathcal{X}_{d+3}^d) & & \\ & \searrow & & \searrow & & \searrow & \\ \cdots & \rightarrow & \mathrm{Mov}(\mathcal{X}_{d+2}^{d+1}) & \rightarrow & \mathrm{Mov}(\mathcal{X}_{d+3}^{d+1}) & \rightarrow & \mathrm{Mov}(\mathcal{X}_{d+4}^{d+1}) \end{array}$$

marked polytopes again

Repeat the same arguments as for the del Pezzo case for $r = d+1,\ d+2,\ d+3$:

- facets of Δ_r^d are Δ_{r-1}^d or Δ_{r-1}^{d-1}
- vertices are associated to $\mathbb{P}^1, \dots, \mathbb{P}^d$
- Δ_r^d are simple (cosimplicial) at vertices $\mathbb{P}^2, \dots, \mathbb{P}^d$ and of special type at $\mathbb{P}^1$
- marked polynomial satisfy some differential equations
- duals of Δ_r^d are known polytopes

problems

- for $r = n+1,\ n+2,\ n+3$ understand the structure of the cone $\mathrm{Mov}(\mathcal{X}_r^n)$, its division by the flopping classes and the action of the Cremona group (e.g. orbits on the set of SQM models)
- do the same for other "exotic" Fano spaces (blow-ups of products of projective spaces which Cox ring is finitely generated): X_7^3, X_8^4, $X_5^{2\times 2}$, $X_6^{2\times 3}$, $X_7^{2\times 4}$.