# Pragmatic 2010 notes, examples, questions
# Part I: resolving singularities

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based on a joint project with Marco Andreatta

# a blow-up

Consider $\mathbb{C}^3$ with coordinates $(x_1, x_2, y)$ and action of $\mathbb{C}^*$ with weights $(1, 1, -1)$, that is $\lambda : \mathbb{C}^* \times \mathbb{C}^3 \to \mathbb{C}^3$ given by the formula

$$\lambda(t)(x_1, x_2, y) = (t \cdot x_1, t \cdot x_2, t^{-1} \cdot y)$$

This is the same as to take $\mathbb{C}[x_1, x_2, y]$ with $\mathbb{Z}$ grading assigning to variables grades $(1, 1, -1)$.

The ring of invariants is

$$\mathbb{C}[x_1, x_2, y]^{\mathbb{C}^*} = \mathbb{C}[yx_1, yx_2] \subset \mathbb{C}[x_1, x_2, y]$$

# a blow-up

Throw away the orbits which converge to 0 when $t \to \infty$, i.e. consider the restriction of the action to $\mathbb{C}^3 \setminus \{x_1 = x_2 = 0\}$ (what will happen if we remove those which converge to 0 when $t \to 0$?)
This set has an affine cover consisting of $U_i = \mathbb{C}^3 \setminus \{x_i = 0\}$ for $i = 1,\ 2$, where

$$U_i = \operatorname{Spec} \mathbb{C}[x_1, x_2, y, x_i^{-1}]$$

We see that

$$\mathbb{C}[x_1, x_2, y, x_1^{-1}]^{\mathbb{C}^*} = \mathbb{C}[x_2/x_1, yx_1]$$

$$\mathbb{C}[x_1, x_2, y, x_2^{-1}]^{\mathbb{C}^*} = \mathbb{C}[x_1/x_2, yx_2]$$

# a blow-up

We have the inclusion

$$\mathbb{C}[x_1, x_2, y]^{\mathbb{C}^*} = \mathbb{C}[yx_1, yx_2] \subset \mathbb{C}[x_1, x_2, y, x_i^{-1}]^{\mathbb{C}^*}$$

If

$$V_i = \operatorname{Spec} \mathbb{C}[x_1, x_2, y, x_i^{-1}]^{\mathbb{C}^*}$$

then $V_1$ and $V_2$ glue over

$$\operatorname{Spec} \mathbb{C}[x_1, x_2, y, (x_1x_2)^{-1}]^{\mathbb{C}^*}$$

to the blow up of $\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[yx_1, yx_2]$ at $(0, 0)$.

# resolution of $\mathbb{A}_1$

Again, take $\mathbb{C}^3$ with coordinates $(x_1, x_2, y)$ and now action with weights $(1, 1, -2)$. The ring of invariants $\mathbb{C}[x_1, x_2, y]^{\mathbb{C}^*}$ is now generated by $z_1 = x_1^2 y,\ z_2 = x_2^2 y,\ z_3 = x_1 x_2 y$ with relation $z_1 z_2 = z_3^2$.

# resolution of $\mathbb{A}_1$

As before throw away the orbits which converge to 0 when $t \to \infty$, i.e. consider the restriction of the action to $\mathbb{C}^3 \setminus \{x_1 = x_2 = 0\}$
Take the cover consisting of $U_i = \mathbb{C}^3 \setminus \{x_i = 0\}$ for $i = 1,\ 2$, where

$$U_i = \operatorname{Spec} \mathbb{C}[x_1, x_2, y, x_i^{-1}]$$

We see that

$$\mathbb{C}[x_1, x_2, y, x_1^{-1}]^{\mathbb{C}^*} = \mathbb{C}[x_2/x_1, yx_1^2]$$

$$\mathbb{C}[x_1, x_2, y, x_2^{-1}]^{\mathbb{C}^*} = \mathbb{C}[x_1/x_2, yx_2^2]$$

# resolution of $\mathbb{A}_1$

Again, we have the inclusion

$$\mathbb{C}[x_1, x_2, y]^{\mathbb{C}^*} = \mathbb{C}[yx_1, yx_2] \subset \mathbb{C}[x_1, x_2, y, x_i^{-1}]^{\mathbb{C}^*}$$

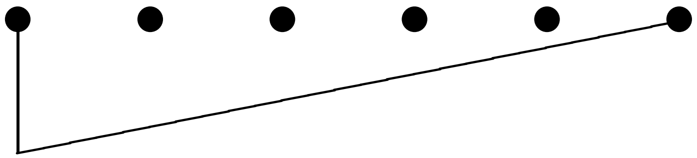
and if $V_i = \operatorname{Spec} \mathbb{C}[x_1, x_2, y, x_i^{-1}]^{\mathbb{C}^*}$ then $V_1$ and $V_2$ glue over $\operatorname{Spec} \mathbb{C}[x_1, x_2, y, (x_1 x_2)^{-1}]^{\mathbb{C}^*}$ to the resolution of $\{z_1 z_2 = z_3^2\} \subset \mathbb{C}^3$.

Excercise: do the same for weights $(1, 1, -n)$.

# toric view

The monomials (characters) invariant with respect to the $\mathbb{C}^*$ action are lattice points in $\widehat{M} = \mathbb{Z}^3$ which are in the kernel $M$ of the map $(a_1, a_2, b) \mapsto a_1 + a_2 - n \cdot b$.

Those element of $M$ with positive coordinates in $\widehat{M}$ (positive octant) form a semigroup spanned by $(i, n-i, 1)$ where $i = 0, \ldots n$.



This way we find out that

$$\mathbb{C}[x_1, x_2, y]^{\mathbb{C}^*} \simeq \mathbb{C}[z_0, \ldots, z_n]/(z_i z_j - z_r z_s \text{ for } i + j = r + s)$$

# toric view

Dually, consider the linear map $\widehat{N} = \mathbb{Z}^3 \to N = \mathbb{Z}^2$ given by the matrix

$$\begin{bmatrix} 0 & n & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

It is surjective and its kernel is $(1, 1, -n)$.

# toric view

Dually, consider the linear map $\widehat{N} = \mathbb{Z}^3 \to N = \mathbb{Z}^2$ given by the matrix

$$\begin{bmatrix} 0 & n & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

It is surjective and its kernel is $(1, 1, -n)$.
Take the fan $\widehat{\Sigma}^+$ consisting of all faces of the positive octant $\widehat{\sigma}^+$. It is mapped to the cone (and of its fan)

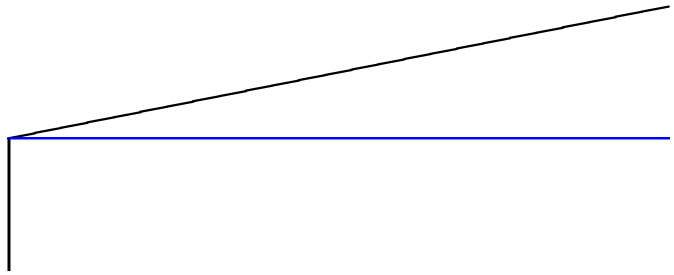
$$\sigma = \mathbb{R}_{\geq 0} \cdot (0, -1) + \mathbb{R}_{\geq 0} \cdot (n, 1)$$

# toric view

Dually, consider the linear map $\widehat{N} = \mathbb{Z}^3 \to N = \mathbb{Z}^2$ given by the matrix

$$\begin{bmatrix} 0 & n & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

It is surjective and its kernel is $(1, 1, -n)$. If we remove "the interior" of $\widehat{\sigma}^+$ and its face $\mathbb{R}_{\geq 0} \cdot (1, 0, 0) + \mathbb{R}_{\geq 0} \cdot (0, 1, 0)$ then the resulting fan maps to the fan obtained by dividing $\sigma$ by the ray $\mathbb{R}_{\geq 0}(1, 0)$:



# toric view, general

Take an exact sequence of lattices

$$0 \longrightarrow P^{\vee} \longrightarrow \widehat{N} \xrightarrow{\pi} N \longrightarrow 0$$

Let $\widehat{M}$ be the dual of $\widehat{N}$ with the positive octant $\widehat{\sigma}^{+}$, we consider the polynomial algebra $\mathbb{C}[\widehat{M} \cap \widehat{\sigma}^{+}]$ with monomials $\chi^{u}$, $u \in \widehat{M}$ and the affine space $\widehat{\mathbb{A}} = \operatorname{Spec} \mathbb{C}[\widehat{M} \cap \widehat{\sigma}^{+}]$.

# toric view, general

Torus $\mathbb{T}_P = P^\vee \otimes \mathbb{C}^*$ acts on $\mathbb{C}[\widehat{M} \cap \widehat{\sigma}^+]$ as follows

$$\lambda \otimes t(\chi^u) = t^{\lambda(u)} \cdot \chi^u$$

The ring of invariants can be computed as follows

$$\mathbb{C}[\widehat{M} \cap \widehat{\sigma}^+]^{\mathbb{T}_P} = \mathbb{C}[M \cap \widehat{\sigma}^+]$$

and it yields a toric variety with big torus $\mathbb{T}_N = N \otimes \mathbb{C}^*$ associated to the cone $\pi(\widehat{\sigma}^+)$

# resolution of $\mathbb{A}_n$

Take a torus $(\mathbb{C}^*)^n$ and let it act on $\mathbb{C}[x_1, y_1, \ldots, y_n, x_2]$ with weights forming the matrix $(n+2) \times n$:

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix}$$

The semigroup $\ker A \cap \widehat{\sigma}^+$ is generated by $(0, 1, 2, \ldots, n+1)$, $(n+1, n, \ldots, 1, 0)$ and $(1, 1, \ldots, 1)$ hence

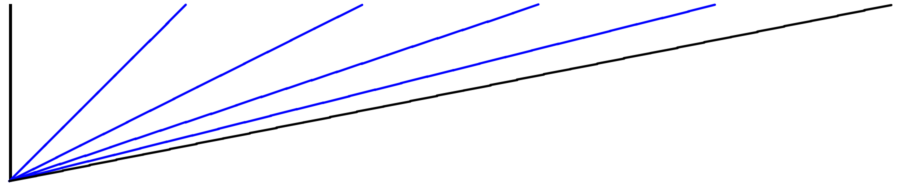
$$\mathbb{C}[x_1, y_1, \ldots, y_n, x_2]^{\mathbb{C}^*} = \mathbb{C}[z_1, z_2, w]/(z_1 z_2 - w^{n+1})$$

# resolution of $\mathbb{A}_n$

The quotient can be described by the surjective map of lattices $\widehat{N} = \mathbb{Z}^{n+2} \to N = \mathbb{Z}^2$ given by the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & n & n+1 \end{bmatrix}$$

Note that $B \cdot A^\top = 0$ or, more precisely, $B^\top$ is the kernel of $A$. Once the images of rays are known then the fan is determined



# $z_1 z_2 = w_1 w_2$, a flop

Now we pass to higher dimensions: consider $\mathbb{C}^*$ action on $\mathbb{C}^4$ with coordinates $x_1, x_2, y_1, y_2$ given by formula

$$\lambda(t)(x_1, x_2, y_1, y_2) = (tx_1, tx_2, t^{-1}y_1, t^{-1}y_2)$$

The ring of invariants is generated by $z_{ij} = x_i y_j$ for $i,\ j = 1,\ 2$ with relation $z_{12}z_{21} = z_{11}z_{22}$ which yields the quadric cone singularity.

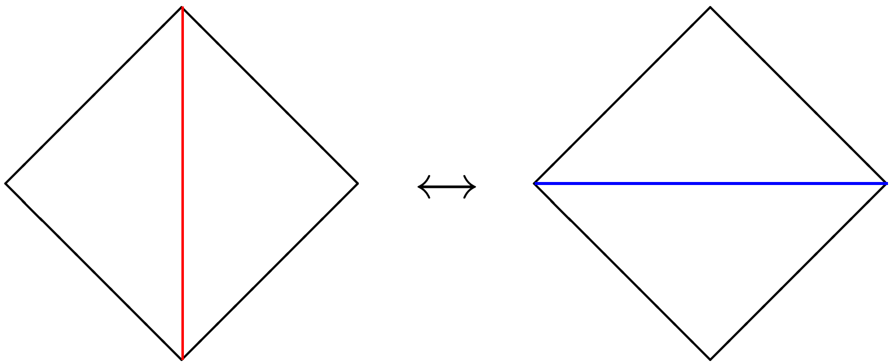
# $z_1 z_2 = w_1 w_2$, a flop

Removing orbits which converge to 0 when $t \to \infty$ yields a quotient with affine covering consisting of $\operatorname{Spec} \mathbb{C}[x_2/x_1, x_1 y_1, x_1 y_2]$ and $\operatorname{Spec} \mathbb{C}[x_1/x_2, x_2 y_1, x_2 y_2]$. These are two copies of $\mathbb{C}^3$ which glue (via the obvious relations) to the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$.

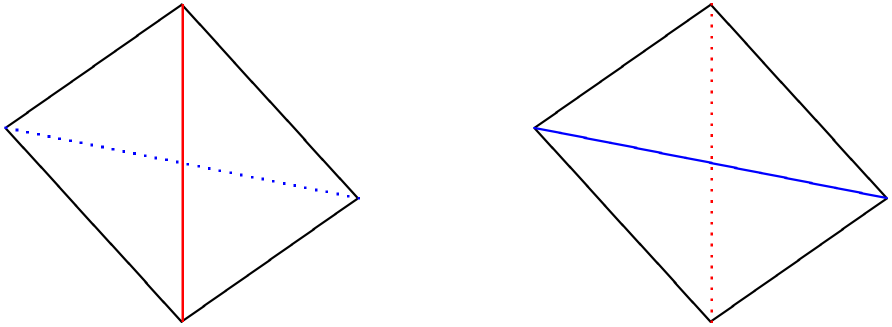
Note that $x$'s and $y$'s are in symmetric position. That is, removing orbits which converge to $\infty$ when $t \to \infty$ yields a quotient with affine covering consisting of $\operatorname{Spec} \mathbb{C}[y_2/y_1, x_1 y_1, x_2 y_1]$ and $\operatorname{Spec} \mathbb{C}[y_1/y_2, x_1 y_2, x_2 y_2]$.

# $z_1 z_2 = w_1 w_2$, a flop

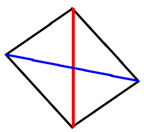
The toric picture is as follows (2 dim section of 3 dim fan):



Which are two projections of the 4 dim cone



# $z_1 z_2 = w_1 w_2$, a flop

Determining which is the front and which is the rear of  is done by a choice of a nonzero vector $u_0$ in the lattice $P = \widehat{M}/M$ which tells you which orbits you want to remove. In general: fix $u_0 \in P = \widehat{M}/M$ take $\widehat{u}_0 \in \widehat{M}$, $\widehat{u_o} \mapsto u_0$ and for $v \in \pi(\sigma^+) \subset N_{\mathbb{R}}$ set $\phi_{u_0}(v) := \sup\{\widehat{u}_0(\widehat{v}) : v \in \sigma^+ \cap \pi^{-1}(v)\}$, assumed $\neq \infty$.

Then $\phi_{u_0}$ is piecewise linear and convex so we can define a fan $\Sigma$ with support on $\pi(\sigma^+)$ by the linear pieces of $\phi_{u_0}$: then $X(\Sigma)$ admits a line bundle associated to this function.

# $z_1 z_2 z_3 = w^2$, more flops

Consider the action of $(\mathbb{C}^*)^3$ on $\mathbb{C}^6$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ given by the matrix of weights

$$A = \begin{bmatrix} 0 & 1 & 1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & -2 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The ring of invariants is generated by $z_1 = x_1^2 y_2 y_3$, $z_2 = x_2^2 y_1 y_2$, $z_3 = x_3^2 y_1 y_2$ and $w = x_1 x_2 x_3 y_1 y_2 y_3$ with the relation $z_1 z_2 z_3 = w^2$.

# $z_1 z_2 z_3 = w^2$, more flops

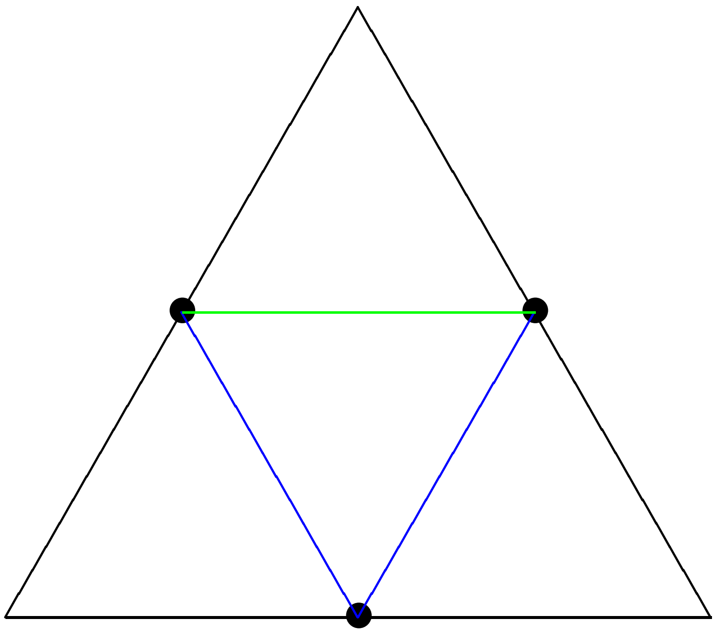
Let us consider a basis $(f_1, f_2, f_3, e_1, e_2, e_3)$ of $\widehat{N}$. The associated map $\pi : \widehat{N} \to N \to 0$ is given by the matrix

$$B = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{bmatrix}$$

where $N$ is the lattice of index 2 in $\mathbb{Z}^3$.

# $z_1 z_2 z_3 = w^2$, more flops

There are several ways of defining the fan of the GIT quotient (here its intersection with a hyperplane containg images of $f_i$'s as vertices of the outer triangle and images of $e_i$'s as $\bullet$)



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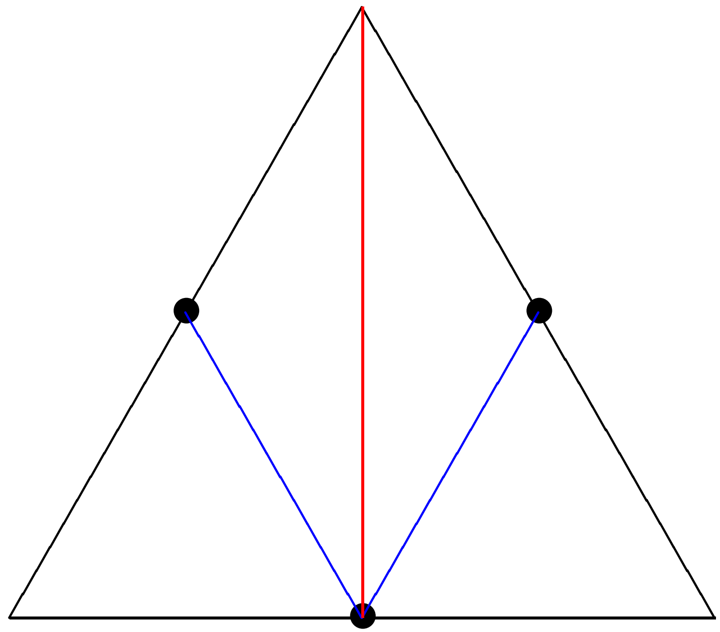
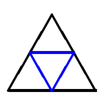
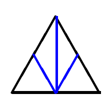


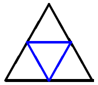
# $z_1 z_2 z_3 = w^2$, more flops

The convexity condition for $u_0 \in P$ says $u_0(2e_i - f_j - f_k) \geq 0$, for $i \neq j \neq k \neq i$, if we want the images of the rays of $\sigma^+$ in the fan $\Sigma$. This defines two dual cones $\mathrm{Mov} \subset P_{\mathbb{R}}$ of admissible $u_0$'s and $\mathrm{Ess} \subset P_{\mathbb{R}}^{\vee}$ of conditions defining them.

Now case  requires $u_0(e_i + e_j - e_k - f_k) > 0$ for $i \neq j \neq k \neq i$ so we can take $\widehat{u}_0 = (0,0,0,1,1,1)$. On the other hand, to get  we take $u_0$ with $u_0(e_2 + e_3 - e_1 - f_1) < 0$, $u_0(2e_2 - f_1 - f_3) > 0$, $u_0(2e_3 - f_1 - f_2) > 0$, hence we use $\widehat{u}_0 = (1.5, 0, 0, 1, 1, 1)$. See Example 1 at

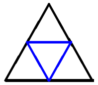
www.mimuw.edu.pl/∼jarekw/java/Pragmatic2010JavaView.html

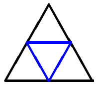
# $z_1z_2z_3 = w^2$, explicit resolution

Look again at  and remember that this is a projection of $\sigma^+$. The faces of $\sigma^+$ which are not seen at this picture are the sets of unstable points, there are six components of them: $\{x_i = y_i = 0\}$ are related to faces $\langle f_i, e_i\rangle$ and $\{x_i = x_j = 0\}$ come from $\langle f_i, f_j\rangle$, for $i \neq j$.

The set of semistable points is covered by four invariant affine sets: $\mathbb{C}^6 \setminus \{x_1x_2x_3 = 0\}$, $\mathbb{C}^6 \setminus \{y_1x_2x_3 = 0\}$, $\mathbb{C}^6 \setminus \{x_1y_2x_3 = 0\}$, $\mathbb{C}^6 \setminus \{x_1x_2y_3 = 0\}$.

# $z_1 z_2 z_3 = w^2$, explicit resolution

The inner cone in  is the image of $\langle e_1, e_2, e_3 \rangle$ hence it is related to the $\mathbb{T}_P$ invariant subalgebra of $\mathbb{C}[x_i^{\pm 1}, y_i]$ which is generated by $w/z_i = (y_i/x_i)x_j x_k$ for $i \neq j \neq k \neq i$. Note that $z_i = (w/z_j) \cdot (w/z_k)$ so the maximal ideal in $\mathbb{C}[z_1, z_2, z_3, w]/(z_1 z_2 z_3 - w^2)$ extends to the ideal of three lines $\{w/z_i = w/z_j = 0\}$.

An outer cone in  is the image of $\langle f_1, e_2, e_3 \rangle$ hence it is related to $\mathbb{T}_P$ invariants of $\mathbb{C}[x_1, x_2^{\pm 1}, x_3, y_1^{\pm 1}, y_2, y_3]$ generated by $w/z_2 = (y_2/x_2)x_1 x_3$, $w/z_3 = (y_3/x_3)x_1 x_2$ and $z_1/w = x_1/(y_1 x_2 x_3)$. What is the extension of the maximal ideal now?

# embedded resolution of $\mathbb{A}_1$

A blow-up of $\mathbb{C}^3$ at the origin can be described in terms of an action of $\mathbb{C}^*$ of $\mathbb{C}^4 = (x_1', x_2', x_3', y')$ with weights $(1, 1, 1, -1)$, the coordinates of $\mathbb{C}^3$ are then $z_i = x_i' y'$. Take zero set of $z_1 z_2 - z_3^2$; its irreducible inverse is $x_1' x_2' - {x_3'}^2$.

# embedded resolution of $\mathbb{A}_1$

Take the map $\mathbb{C}[x'_1, x'_2, x'_3, y'] \longrightarrow \mathbb{C}[x_1, x_2, y]$, where the latter variables have grades $(1, 1, -2)$ such that $(x'_1, x'_2, x'_3, y') \mapsto (x_1^2, x_2^2, x_1 x_2, y)$. Its image is the even graded part of $\mathbb{C}[x_1, x_2, y]$ and its kernel is $(x'_1 x'_2 - {x'_3}^2)$ thus $\mathbb{C}[x'_1, x'_2, x'_3, y']/(x'_1 x'_2 - {x'_3}^2) \subset \mathbb{C}[x_1, x_2, y]$ and both have the same $\mathrm{Proj}$.

However the former one is **not** quite what we want!

We want to get a functorial object, the Cox ring.

# the Cox ring, 1st view

Let $V \subset \mathbb{C}^n$ be an affine variety with coordinate ring $A$. Suppose that it admits a resolution of singularities $\widehat{V} \to V$ and assume that $\mathrm{Pic}(\widehat{v}/V)$ is a lattice with a basis $L_1, \ldots, L_r$. The Cox ring of $\widehat{V} \to V$ is an $A$-algebra

$$\widehat{A} = \bigoplus \Gamma(\mathcal{O}(m_1 L_1 + \cdots + m_r L_r))$$

with $\mathbb{Z}^r$ grading. We will consider good singularities for which $\widehat{V} \to V$ will be crepant.

# the Cox ring, 1st view

In our situation (take this as an assumption, if you want):

- $\widehat{A}$ is finitely generated $\mathbb{C}$-algebra with $(\mathbb{C}^*)^r$ action
- $A$ is the ring of invariants of $\widehat{A}$ under the induced $(\mathbb{C}^*)^r$ action
- all crepant (good) resolutions of $V$ are GIT quotients of $\operatorname{Spec} \widehat{A}$
- the toric case works nicely: take $\sigma \subset N_{\mathbb{R}}$ a (pointed) cone with generators of rays lying on an affine hyperplane, no lattice points lying below that hyperplane and the points on that hyperlane being vertices of a unimodular triangulation

# Atiyah flop, Mukai flop

Take $\mathbb{C}^*$ action on $\mathbb{C}^r \times \mathbb{C}^s$ with coordinates $(x_i, y_j)$ and weights $1$ for $x_i$'s and $-1$ for $y_j$'s. The quotient is toric singularity associated to cone spanned by $s$ vectors $e_i$ and $r$ vectors $f_j$ in the lattice of rank $r+s-1$ with one relation $\sum e_i = \sum f_j$. If $r = s$ this admits two crepant resolutions associated to two unimodular triangulations: one in which we omit consecutive $e_i$'s, the other in which we omit $f_j$'s. Find the affine pieces of covering, they should be of type $\operatorname{Spec} \mathbb{C}[x_i/x_1, x_1 y_j]$.

Verify it by looking at the cone over Segre embedding of $\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

# Atiyah flop, Mukai flop

Consider the quadric hypersurface $\{\sum_i x_i y_i = 0\} \subset (\mathbb{C}^r)^2$ which is invariant with respect to the $\mathbb{C}^*$ action. Take the symplectic form $\omega = \sum_i (dx_i \wedge dy_i)$ and evaluate it on the vector field $\sum_i (x_i \partial_{x_i} - y_i \partial_{y_i})$ which is tangent to the $\mathbb{C}^*$ action. The result is $\sum_i (x_i dy_i + y_i dx_i)$, the derivative of the defining equation.

This implies that $\omega$ descends to a symplectic form on the quotient(s). See it in local coordinates.

Another view: standard symplectic form on the cotangent bundle od $\mathbb{P}^{r-1}$.

This way for $r > 1$ we get the only isolated singularity with symplectic resolution.

# more symplectic resolutions

See Example 1A at

www.mimuw.edu.pl/∼jarekw/java/Pragmatic2010JavaView.html

# resolution of $\mathbb{D}_4$

Take a hyperplane section of the singularity $z_1 z_2 z_3 = w^2$ defined by the relation $z_1 + z_2 + z_3 = 0$. The lift-up of this hyperplane to the resolution discussed above has one singular point of type $\mathbb{A}_1$ because

$$z_1 + z_2 + z_3 = \frac{w}{z_1} \cdot \frac{w}{z_2} + \frac{w}{z_1} \cdot \frac{w}{z_3} + \frac{w}{z_2} \cdot \frac{w}{z_3}$$

(Check that there are no other singular points in other affine sets of the covering.)

# resolution of $\mathbb{D}_4$

We can do the embedded resolution but this will not yield the Cox ring.

# problems

For the start, let us consider two classes of quotient singularities:

- surface Du Val or $\mathbb{A}-\mathbb{D}-\mathbb{E}$ singularities:
  - $x^{n+1}+y^2+z^2=0$
  - $x^{n-1}+xy^2+z^2=0$
  - $x^4+y^3+z^2=0$, $x^3y+y^3+z^2=0$, $x^5+y^3+z^2=0$
- 4-dimensional quotient symplectic singularities: wreath product of $\mathbb{A}-\mathbb{D}-\mathbb{E}$ and $\mathbb{Z}_2$

# problems

- find the Cox ring of these singularities
- find the structure of of $\mathrm{Mov}$ and its division by flopping classes