Chapter 6

Quotients

6.1 Categorical and geometric quotients

Let G be an affine algebraic group acting (rationally, as always) on an algebraic variety X over an algebraically closed field k. We would like to define the quotient variety X/G whose points are orbits. As we explained in Chapter 1 this is a hopeless task due to the existence of nonclosed orbits. So we need to modify the definition of X/G; for this we look first at the categorical notion of a quotient object with respect to an equivalence relation.

Let (X, R) be a set together with an equivalence relation $R \subset X \times X$. The canonical map $p : X \to X/R$ has the universal property with respect to all maps $f : X \to Y$ such that $R \subset X \times_Y X = (f \times f)^{-1}(\Delta_Y)$. Also we have $R = X \times_{X/R} X = (p \times p)^{-1}(\Delta_{X/R})$. This equality expresses the property that the fibres of the map p are the equivalence classes. Let us express this in categorical language. Let $\mathcal{C}$ be any category with fibred products. We define an *equivalence relation* on an object X as a subobject $R \subset X \times X$ (or more generally just a morphism $R \to X \times X$) satisfying the obvious axioms (expressed by means of commutative diagrams). Then we define a quotient X/R as an object in $\mathcal{C}$ for which there is a morphism $p : X \to X/R$ having the universal property with respect to morphisms $X \to Y$ such that $R \to X \times X$ factors through a morphism $R \to X \times_Y X$. By definition there is a canonical morphism

$$R \to X \times_{X/R} X. \tag{6.1}$$

Note that, in general, there is no reason to expect that in general the morphism (6.1) will be an isomorphism or an epimorphism.

Let $\sigma : G \times X \to X$ be an algebraic action. We say that the pair (X, σ) is a *G-variety* and often drop σ from the notation. Let $\Psi : G \times X \to X \times X$ be the morphism (σ, pr_2). This morphism should be thought of as an equivalence relation on X defined by the action. A G-equivariant morphism of G-varieties corresponds to a morphism of sets with an equivalence relation. The definition of a G-equivariant morphism $f : X \to Y$ can be rephrased by saying that the map Ψ factors through the natural morphism $X \times_Y X \to X \times X$; this corresponds to the property $(f, f)(R) \subset \Delta$. This suggests the following definition.

Definition. A *categorical quotient* of a G-variety X is a G-invariant morphism $p : X \to Y$ such that for any G-invariant morphism $g : X \to Z$ there exists a unique morphism $\bar{g} : Y \to Z$ satisfying $\bar{g} \circ p = g$. A categorical quotient is called a *geometric quotient* if the image of the morphism Ψ equals $X \times_Y X$. We shall denote the categorical quotient (resp. geometric quotient) by $p : X \to X/\!\!/G$ (resp. $p : X \to X/G$). It is defined uniquely up to isomorphism.

A different approach to defining a geometric quotient is as follows. We know how to define a geometric quotient as a set; we next discuss topological spaces. We put the structure of a topological space on X/G so that the canonical projection $p : X \to X/G$ is continuous. The weakest topology on X/G for which this will be true is the topology in which a subset $U \subset X/R$ is open if and only if $p^{-1}(U)$ is open. Then we examine ringed spaces, whose definition is given in terms of choosing a class of functions on X (e.g. regular functions, smooth functions, analytic functions). If $\phi \in \mathcal{O}(U)$ is a function on $U \subset X/R$, then the composition $p^*(\phi) = \phi \circ p$ must be a function on $p^{-1}(U)$. It is obviously a G-invariant function. Using this remark we can define the structure of a ringed space on X/R by setting $\mathcal{O}(U) = \mathcal{O}(p^{-1}(U))^G$. This makes $p : X \to X/R$ a categorical quotient in the category of ringed spaces. Finally, we want the fibres of p to be orbits; this is the condition that the morphism (6.1) is an isomorphism.

Definition. A *good geometric quotient* of a G-variety X is a G-invariant morphism $p : X \to Y$ satisfying the following properties:

(i) p is surjective;

(ii) for any open subset U of Y, the inverse image $p^{-1}(U)$ is open if and only if U is open;

(iii) for any open subset U of Y, the natural homomorphism $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}(p^{-1}(U))^G$ of G-invariant functions;

(iv) the image of $\Psi : G \times X \to X \times X$ is equal to $X \times_Y X$.

Proposition 6.1. *A good geometric quotient is a categorical quotient.*

Proof. Let $q : X \to Z$ be a G-invariant morphism. Pick any affine open cover $\{V_i\}_{i\in I}$ of Z. For any V_i the inverse image $q^{-1}(V_i)$ will be an open G-invariant subset of X. Then we have the obvious inclusion $q^{-1}(V_i) \subset p^{-1}(U_i)$, where $U_i = p(q^{-1}(V_i))$. Comparing the fibres over points $y \in Y$ and using property (iv) (which says that the fibres of p are orbits), we conclude that in fact $q^{-1}(V_i) = p^{-1}(U_i)$. By property (ii), U_i is open in Y. Since p is surjective we get an open cover $\{U_i\}_{i\in I}$ of Y. The map $q^{-1}(V_i) \to V_i$ is defined by a homomorphism

$$\alpha_i : \mathcal{O}(V_i) \to \mathcal{O}(q^{-1}(V_i)) = \mathcal{O}(p^{-1}(U_i)).$$

Since q is a G-invariant morphism, the image of α_i is contained in the subring $\mathcal{O}(p^{-1}(U_i))^G$ of $\mathcal{O}(U_i)$. This defines a unique homomorphism $\mathcal{O}(V_i) \to \mathcal{O}(U_i)$, and hence a unique morphism $\bar{q}_i : U_i \to V_i$ (because V_i is affine). It is immediately checked that the maps $\bar{p}_i$ agree on the intersections $U_i \cap U_j$, and hence define a unique morphism $\bar{q} : Y \to Z$ satisfying $q = \bar{q} \circ p$. □

Proposition 6.2. *Let $p : X \to Y$ be a G-equivariant morphism satisfying the following properties:*

(i) for any open subset U of Y, the homomorphism of rings $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}(p^{-1}(U))^G$ of G-invariant functions;

(ii) if W is a closed G-invariant subset of X then $p(W)$ is a closed subset of Y;

(iii) if W_1, W_2 are closed invariant subsets of X with $W_1 \cap W_2 = \emptyset$, then $p(W_1) \cap p(W_2) = \emptyset$.

Under these conditions p is a categorical quotient. It is a good geometric quotient if additionally

(iv) the image of $\Psi : G \times X \to X \times X$ is equal to $X \times_Y X$.

Conversely, a good geometric quotient satisfies properties (i)–(iv).

Proof. This is similar to the previous proof. With the same notation, let

$$W_i = X \setminus q^{-1}(V_i).$$

This is a closed G-invariant subset of X, hence, by (ii), $U_i = Y \setminus p(W_i)$ is an open subset of Y. Clearly, $p^{-1}(U_i) \subset q^{-1}(V_i)$. Since $\bigcap_i W_i = \emptyset$, by (iii) we have $\bigcap_i p(W_i) = \emptyset$, hence $Y = \bigcup_i U_i$. Now composing the homomorphisms $\alpha_i : \mathcal{O}(V_i) \to \mathcal{O}(q^{-1}(V_i))^G$ with the restriction homomorphism $\mathcal{O}(q^{-1}(V_i))^G \to \mathcal{O}(p^{-1}(U_i))^G = \mathcal{O}(U_i)$ we get a homomorphism $\mathcal{O}(V_i) \to \mathcal{O}(U_i)$. Since V_i is affine this defines a morphism $U_i \to V_i$ whose composition with $p : p^{-1}(U_i) \to U_i$ is the map $q : p^{-1}(U_i) \to V_i$. Gluing together these morphisms we construct $Y \to Z$ as in the proof of Proposition 6.1. This shows that Y is a categorical quotient.

Let us check that under condition (iv) $p : X \to Y$ is a good geometric quotient. First we see that p is surjective. Indeed, (i) implies that p is dominant and (iii) implies that $p(X)$ is closed. Also property (ii) implies property (ii) of the definition of a good geometric quotient. In fact, if $p^{-1}(U)$ is open, then $X \setminus p^{-1}(U)$ is closed and G-invariant. Since p is surjective, its image is equal to $Y - U$ and is closed. Therefore U is open. This checks the definition.

Conversely, assume $p : X \to Y$ is a good geometric quotient. Properties (i) and (iv) follow from the definition. Let us check properties (ii) and (iii). The set $U = X \setminus W$ is open and invariant. Since the fibres of p are orbits, $U = p^{-1}(p(U))$ and hence $p(U)$ is open. For the same reason, $W = p^{-1}(p(W))$ and hence $p(W) = Y \setminus p(W)$ is closed. Furthermore, $W_1 \cap W_2 = p^{-1}(p(W_1)) \cap p^{-1}(p(W_2)) = p^{-1}(p(W_1) \cap p(W_2))$. This checks property (iii). □

Corollary 6.1. *Under the assumptions from the preceding Proposition, the map $p : X \to Y$ satisfies the following properties:*

(i) two points $x, x' \in X$ have the same image in Y if and only if $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$;

(ii) for each $y \in Y$ the fibre $p^{-1}(y)$ contains a unique closed orbit.

Proof. In fact, the closures of orbits are closed G-invariant subsets in X. So if $\overline{G \cdot x} \cap \overline{G \cdot x'} = \emptyset$, $p(\overline{G \cdot x}) \cap p(\overline{G \cdot x'}) = \emptyset$. But both sets contain the point $p(x) = p(x')$. Conversely, if $p(x) \neq p(x')$, we get that $G \cdot x$ and $G \cdot x'$ lie in different fibres. Since the fibres are closed subsets, $\overline{G \cdot x}$ and $\overline{G \cdot x'}$ lie in different fibres, and hence they are disjoint. This proves (i). To prove (ii) we notice that by (i) two closed orbits in the same fibre must have nonempty intersection, but this is absurd. Since each fibre contains at least one closed orbit, we are done. □

Definition. A categorical quotient satisfying properties (i), (ii) and (iii) from Proposition 6.2 is called a *good categorical quotient.*

Remarks 6.1. 1. Note that condition (ii) in the definition of a good geometric quotient is satisfied if we require

(ii)' for any closed G-invariant subset Z of X, the image $p(Z)$ is closed. Also, together with condition (iii) this implies the surjectivity of the factor map p. In fact, condition (iii) ensures that the map p is dominant, i.e., its image is dense in Y. But by (ii)', the image of p must be closed.

2. Suppose X is an irreducible normal G-variety over an algebraically closed field of characteristic 0, and $p : X \to Y$ is a surjective G-invariant morphism such that Y is normal and each fibre over any point $y \in Y$ is an orbit. Then $p : X \to Y$ is a geometric quotient. The proof is rather technical and we omit it (see [74], Proposition 0.2).

3. The definitions of categorical and geometric quotients are obviously "local" in the following sense: If $p : X \to Y$ is a G-equivariant morphism, and $\{U_i\}$ is an open cover of Y with the property that each $p_i : p^{-1}(U_i) \to U_i$ is a categorical (resp. geometric) quotient, then p is a categorical (resp. geometric) quotient.

6.2 Examples

Let us give some examples.

Example 6.1. Let G be a finite group considered as an algebraic group over a field k. Assume that X is quasi-projective. Then the geometric quotient X/G always exists. In fact, assume first that X is affine. By Theorem 3.1, the algebra $\mathcal{O}(X)^G$ is finitely generated over k. Let Y be an affine algebraic variety with $\mathcal{O}(Y) = \mathcal{O}(X)^G$. By the theorems on lifting of ideals in integral extensions, the map $p : X \to Y = X/G$ satisfies properties (ii) and (iii) from Proposition 6.2. Also, the group G acts transitively on the set of prime ideals in $\mathcal{O}(X)$ which lie over a fixed prime ideal of $\mathcal{O}(Y)$ (see, for example, [9], Chapter V, §2, Theorem 2). This shows that $\Psi : G \times X \to X \times_Y X$ is an isomorphism.

Now let $X \subset \mathbb{P}^n$ be quasi-projective but not necessarily affine. Let $\bar{X}$ be the closure of X. Let $O \subset X$ be an orbit and let F be a homogeneous polynomial vanishing on $\bar{X} \setminus X$ but not vanishing at any point of O. Thus O is contained in an ffine subset $U = \bar{X} \setminus V(F)$. Recall that the complement of a hypersurface in a projective space is an open affine subset. This implies that U, being closed in an affine set, is affine. Let $U(O) = \bigcap_{g \in G} (g \cdot U)$. This is an open G-invariant affine subset of X containing O. By letting O vary, we get an open affine G-invariant covering $\{U_i\}$ of X. We already know that each quotient $p_i : U_i \to U_i/G = V_i$

exists. We will glue the V_i together to obtain the geometric quotient $p : X \to X/G$ (we refer to the gluing construction of algebric varieties in section 8.2). To do this we observe first that $U_i \cap U_j$ is affine and $U_i \cap U_j/G$ is open in V_i and V_j; this follows from considering the affine case. Thus we can glue V_i and V_j together along the open subset $V_{ij} = U_i \cap U_j/G$; we do this for all i and j. The resulting algebraic variety Y is separated. In fact we use that in the affine situation

$$(X_1 \times X_2)/(G_1 \times G_2) \cong X_1/G_1 \times X_2/G_2,$$

where $G_1 \times G_2$ acts on $X_1 \times X_2$ by the Cartesian product of the actions. Thus the image of $\Delta_X \cap (U_i \times U_j)$ in $(U_i \times U_j)/(G \times G) \cong U_i/G \times U_j/G$ is closed, and, as is easy to see, coincides with $\Delta_Y \cap (V_i \times V_j)$. This shows that Δ_Y is closed. It remains to prove that X/G is quasi-projective; we shall do this later. Note that, if X is not a quasi-projective algebraic variety, X/G may not exist in the category of algebraic varieties even in the simplest case when G is of order 2. The first example of such an action was constructed by M. Nagata ([76]) in 1956 and later a simpler construction was given by H. Hironaka (unpublished). However, if we assume that each orbit is contained in a G-invariant open affine subset, the previous construction works and X/G exists.

Example 6.2. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a finitely generated k-algebra with a geometric grading (see Example 3.1). Consider the corresponding action of $\mathbb{G}_m$ on $X = \mathrm{Spm}(A)$. Let x_0 be the vertex of X defined by the maximal ideal $\bigoplus_{i=1}^{\infty} A_i$. Then the open subset $X' = X \setminus x_0$ is invariant and the geometric quotient $X/\!\!/\mathbb{G}_m$ exists and is isomorphic to the projective variety $\mathrm{Projm}(A)$. We leave the details to the reader.

Example 6.3. Let H be a closed subgroup of an affine algebraic group G and G/H be the coset space (see Example 3.3). The canonical projection $G \to G/H$ is a good geometric quotient. We omit the proof, referring the reader to [52], IV, 12, where all conditions of the definition are verified.

Let us show now that the categorical quotient of an affine variety always exists. We will need the following lemma.

Lemma 6.1. *Let X be an affine G-variety, and let Z_1 and Z_2 be two closed G-invariant subsets with $Z_1 \cap Z_2 = \emptyset$. Assume G is geometrically reductive. Then there exists a G-invariant function $\phi \in \mathcal{O}(X)^G$ such that $\phi(Z_1) = 0, \phi(Z_2) = 1$.*

Proof. First choose a function $\varphi \in \mathcal{O}(X)$, not necessarily G-invariant, such that $\varphi(Z_1) = 0, \varphi(Z_2) = 1$. This is easy: since the sum of the ideals defining Z_1 and

Z_2 is the unit ideal, we can find a function $\alpha \in I(Z_1)$ and a function $\beta \in I(Z_2)$ such that $1 = \alpha + \beta$. Then we take $\varphi = \alpha$. Let W be the linear subspace of $\mathcal{O}(X)$ spanned by the translates $g^*(\varphi), g \in G$. We know that it is finite-dimensional (Lemma 3.4); let $\varphi_1, \ldots, \varphi_n$ be a basis. Consider the map $f : X \to \mathbb{A}^n$ defined by these functions. Clearly, $f(Z_1) = (0, \ldots, 0), f(Z_2) = (1, \ldots, 1)$. The group G acts linearly on the affine space, defining a linear representation. By definition of geometrically reductive groups, we can find a nonconstant G-invariant homogeneous polynomial $F \in k[Z_1, \ldots, Z_n]$ such that $F(1, \ldots, 1) = 1$. Then $\phi = f^*(F) = F(\varphi_1, \ldots, \varphi_n)$ satisfies the assertion of the lemma. □

Now we are ready to prove the following main result of this chapter:

Theorem 6.1. *Let G be a geometrically reductive group acting on an affine variety X. Then the subalgebra $\mathcal{O}(X)^G$ is finitely generated over k, and the canonical morphism $p : X \to Y = \mathrm{Spm}(\mathcal{O}(X)^G)$ is a good categorical quotient.*

Proof. The first statement is Nagata's Theorem proven in Chapter 3. To show that p is a good categorical quotient, we apply Proposition 6.2. First of all, property (i) easily follows from the fact that taking invariants commutes with localizations. More precisely, if $f \in \mathcal{O}(X)^G$, then $(\mathcal{O}(X)_f)^G = (\mathcal{O}(X)^G)_f$; this is easy and we skip the proof. Next let Z be a closed G-invariant subset of X. Suppose $p(Z)$ is not closed. Let $y \in \overline{p(Z)} \setminus p(Z)$. Then $W_1 = Z$ and $W_2 = p^{-1}(y)$ are two closed G-invariant subsets of X with empty intersection. By the preceding Lemma, there exists a function $\phi \in \mathcal{O}(X)^G$ such that $\phi(Z) = 0, \phi(p^{-1}(y)) = 1$. Since $\phi = p^*(\varphi)$ for some $\varphi \in \mathcal{O}(Y)$, we obtain $\varphi(p(Z)) = 0, \varphi(y) = 1$. But this is absurd since y belongs to the closure of $p(Z)$. This verifies condition (ii). Now let Z_1 and Z_2 be two disjoint G-invariant closed subsets of X. As above we find a function $\varphi \in \mathcal{O}(Y)$ with $\varphi(p(W_1)) = 0, \varphi(p(W_2)) = 1$. This obviously implies that $p(Z_1) \cap p(Z_2) = \emptyset$. This verifies (iii). □

Example 6.4. We have already discussed this example in Chapter 1. Let $G = \mathrm{GL}_n$ act on itself by the adjoint action, i.e. $g \cdot x = gxg^{-1}$. For each matrix $g \in \mathrm{GL}_n$ we consider the characteristic polynomial

$$\det(g - tI_n) = (-t)^n + c_1(g)(-t)^{n-1} + \cdots + c_n(g).$$

Define a regular G-equivariant map $c : \mathrm{GL}_n \to \mathbb{A}^n$ by the formula $c(g) = (c_1(g), \ldots, c_n(g))$. We claim that this is a categorical quotient. To check this it is enough to verify that $\mathcal{O}(G)^G = k[c_1, \ldots, c_n)] \cong k[Z_1, \ldots, Z_n]$; this is what we did in Chapter 1. It is clear that the fibre of c does not consist of one orbit, so the quotient is not a geometric quotient.

6.3 Rational quotients

We know that neither X/G nor $X/\!\!/G$ exists in general. So a natural problem is to find all possible open subsets of X for which the categorical or geometric quotient exists. Geometric invariant theory gives a solution to this problem when we additionally assume that the quotient is a quasi-projective algebraic variety.

Let us first show that any open subset U for which a geometric quotient U/G exists must be contained in a certain open subset X_{reg}. We will assume in the sequel that G is connected. Otherwise, we consider its connected component G^o containing the identity element. It is a normal closed subgroup of G and the quotient G/G^o is a finite group. It is easy to see (see Exercise 6.11) that we can divide by G in two steps: first divide by G^o, and then divide the quotient by the finite group G/G^o.

For any point $x \in X$ we have a regular map

$$\sigma_x : G \to X, \quad g \mapsto \sigma(g, x) := g \cdot x.$$

Clearly the image of this map is the G-orbit $\mathrm{O}(x)$ of the point x.

The set theoretical fibre of this map at a point x is denoted by G_x and is called the *isotropy subgroup* of x in the action σ. It is a closed subgroup of G, hence an affine algebraic group. If $\mathrm{char}(k) = 0$, the set theoretical fibre of σ_x coincides with the scheme theoretical fibre (or, in other words, the latter is a reduced closed subscheme of G). We are not going to prove this; to do so we would have to go into the theory of *group schemes* and prove the fundamental result of the theory that every group scheme over a field of characteristic zero is reduced.

Since all fibres of σ_x over points in $\mathrm{O}(x)$ are isomorphic (they are conjugate subgroups of G_x), the theorem on the dimension of fibres (see [103]) gives

$$\dim \mathrm{O}(x) = \dim G - \dim G_x. \tag{6.2}$$

If $\overline{\mathrm{O}(x)} \neq \mathrm{O}(x)$, the complement $\overline{\mathrm{O}(x)} \setminus \mathrm{O}(x)$ is a proper closed subset of $\overline{\mathrm{O}(x)}$, hence its dimension is strictly less than $\dim \overline{\mathrm{O}(x)}$. Take any $y \in \overline{\mathrm{O}(x)} \setminus \mathrm{O}(x)$ and consider its orbit $\mathrm{O}(y)$. Since $\dim \overline{\mathrm{O}(y)} < \dim \overline{\mathrm{O}(x)}$, applying (6.2) to y we see that

$$\dim G_x < \dim G_y. \tag{6.3}$$

Let

$$I = \Phi^{-1}(\Delta_X) = \{(g, x) \in G \times X : \sigma(g, x) = x\}.$$

This is a closed subset of $G \times X$. Consider the second projection $\mathrm{pr}_2 : I \to X$. Its fibre over a point $x \in X$ is isomorphic to the isotropy subgroup G_x. By the

theorem on the dimension of fibres applied to pr_2, there exists an open subset X_{reg} of X such that $\dim G_x = d$ for all $x \in X_{\mathrm{reg}}$ and $\dim G_x > d$ for all $x \notin X_{\mathrm{reg}}$.

Applying (6.2) we obtain that for any $x \in X_{\mathrm{reg}}$ the orbit $\mathrm{O}(x)$ is closed in X_{reg} and has dimension equal to $\dim G - d$. Also, any other orbit in X has dimension strictly less than $\dim G - d$. Let U be any G-invariant open subset of X for which a geometric quotient $U \to U/G$ exists. We assume that X is irreducible. So $p_U : U \cap X_{\mathrm{reg}} \neq \emptyset$ and hence some of the orbits in U must be of dimension $\dim G - d$. By the theorem on dimension of fibres all fibres of p_U have dimension greater than or equal to $\dim G - d$ and hence all fibres of p_U have dimension equal to $\dim G - d$. Therefore they are contained in X_{reg} and hence $U \subset X_{\mathrm{reg}}$.

Thus we get a necessary condition for the existence of U/G: U must be an open subset of X_{reg}.

Theorem 6.2. *(M. Rosenlicht) Assume X is irreducible. Then X_{reg} contains an open subset U such that a good geometric quotient $U \to U/G$ exists with quasi-projective U/G. The field of rational functions on U/G is isomorphic to the subfield $k(X)^G$ of G-invariant rational functions on X.*

Proof. The proof is easy if we assume additionally that G is geometrically reductive and X is affine. Let Y be an algebraic variety with field of rational functions isomorphic to $k(X)^G$; such a Y always exists since $k(X)^G$ is of finite transcendence degree over k. Consider the rational dominant map $X_{\mathrm{reg}} \to Y$ defined by the inclusion of the fields $k(X)^G \subset k(X)$. By deleting some subset from X_{reg} we find a G-invariant open subset $U \subset X_{\mathrm{reg}}$ and a regular map from $f : U \to Y$. Replacing Y by an open subset we may assume that f is surjective. This is condition (i) from the definition of a good geometric quotient. For any open subset $V \subset U$ we have an inclusion $\mathcal{O}(V) \subset k(Y) = k(X)^G$. Since $f^*(\mathcal{O}(V)) \subset \mathcal{O}(f^{-1}(V))$ we see that $f^*(\mathcal{O}(V)) \subset \mathcal{O}(f^{-1}(V))^G$. Conversely $\mathcal{O}(f^{-1}(V))^G \subset k(X)^G = k(Y)$ and hence $\mathcal{O}(f^{-1}(V))^G \subset \mathcal{O}(V)$. Thus we have checked condition (i) of Proposition 6.2. Since U is G-invariant, the fibres of f are unions of orbits. Since any orbit in X_{reg} is closed in X_{reg}, it is closed in U. By Lemma 6.1 we can separate closed invariant subsets by functions from $\mathcal{O}(V)$. This shows that the fibres of f are orbits. This checks condition (iv). The conditions (ii) and (iii) of Proposition 6.2 are checked by using the argument from the proof of Theorem 6.1.

Let us give an idea for the proof in the general case. For the details we refer to the original paper of Rosenlicht ([94]; see also [90], 2.3). Since we do not assume that X is affine, even if G is geometrically reductive we cannot separate the closed orbits contained in the fibres of the map $f : U \to Y$. Consider the generic fibre of f as an algebraic variety U_η over the field $K = k(Y) = k(X)^G$.

Let $\bar{K}$ be the algebraic closure of K. The group $G(\bar{K})$ acts on $U_\eta(\bar{K})$ and the field of invariant rational functions is isomorphic to $\bar{K}$. All orbits of $G(\bar{K})$ have the same dimension. Suppose that a group G acts on an irreducible quasi-projective variety $X \subset \mathbb{P}^N$ such that all orbits are of the same dimension and closed. We define a map from X to the Chow variety parametrizing closed subsets of $\mathbb{P}^n$ of the same dimension d (see [74], Chapter 4, §6) by assigning to a point $x \in X$ the closure of the orbit $G \cdot x$. If the image is of positive dimension, we can construct a nonconstant invariant function on X by taking the inverse image of a rational function on the image. Otherwise the image is one point, and we obtain that X consists of one orbit. Applying this argument to $U_\eta(\bar{K})$ we see that it consists of one orbit. This implies that there is an open subset of Y such that each fibre consists of one orbit. Again deleting a closed subset from Y we may assume that Y is nonsingular. Since the dimension of all orbits is the same, the morphism f is open; this is called *Chevalley's criterion* (see [6], p. 44). This verifies condition (ii) of the definition of a good geometric quotient. The remaining conditions have been checked already. □

Corollary 6.2. *The transcendence degree of* $k(X)^G$ *is equal to* $\dim X - \dim G + d$, *where* $d = \min_{x \in X}\{\dim G_x\}$.

Any model of $k(X)^G$ is called a *rational quotient* of X by G. We see that X contains an open subset such that a good geometric quotient U/G exists and coincides with a rational quotient.

Bibliographical notes

The notions of a categorical and geometric quotients are originally due to Mumford ([74]). Many books discuss different versions of these notions (see [63], [81]). Many interesting results about the structure of fibres of the quotient maps have been omitted; we refer to [90] for a survey of these results.

Exercises

6.1 Let $\mathbb{G}_a$ act on $\mathbb{A}^2$ by the formula $t \cdot (z_1, z_2) = (z_1, z_2 + tz_1)$. Consider the map $\mathbb{A}^2 \to \mathbb{A}^1$, $(z_1, z_2) \mapsto z_1$. Is it a categorical quotient? If so, is it a geometric quotient?

6.2 Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ by the formula $t \cdot (z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_n)$ for some positive integers $q_1, \ldots, q_n$ coprime to $\mathrm{char}(k)$. Let $A = k[T_1, \ldots, T_n]$ with the corresponding geometric grading defined by the action. Show that the geometric quotient $\Big(\mathbb{A}^n \setminus \{0\}\Big)/\mathbb{G}_m$ (see Example 6.2) is isomorphic to a quotient of $\mathbb{P}^{n-1}$ by a finite group.

6.3 Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded finitely generated k-algebra, and $A^{(e)} = \bigoplus_{i \in \mathbb{Z}} A_{ei}$. Show that, if e is coprime to $\mathrm{char}(k)$, $A^{(e)} = A^G$, where G is a cyclic group of order e.

6.4 Construct a counterexample to Lemma 6.1 when $G = \mathbb{G}_a$ is the additive group.

6.5 In the notation of Nagata's Theorem show that for any open subset U of Y, the restriction map $p^{-1}(U) \to U$ is a categorical quotient with respect to the induced action of G.

6.6 Describe the orbits and the fibres of the categorical quotient from Example 6.4 when $n = 2$.

6.7 Show that the categorical quotient of $\mathrm{Pol}_3(k^2)$ by SL_2 is isomorphic to $\mathbb{A}^1$. Describe the orbits and the fibres of the categorical quotient.

6.8 Let G act on an irreducible affine variety X and let $f : X \to Y$ be a G-invariant morphism to a normal affine variety. Assume that $\mathrm{codim}(Y \setminus f(X), Y) \geq 2$ and that there exists an open subset U of Y such that for all $y \in U$ the fibre $f^{-1}(y)$ contains a dense orbit. Show that $Y \cong X/\!\!/G$.

6.9 Let G be a finite group of automorphisms of an irreducible algebraic variety. Prove that $k(X/G) = k(X)^G$.

6.10 Show by example that in general the field of fractions $Q(A^G)$ of the ring of invariants A^G is not equal to $Q(A)^G$. Prove that $Q(A^G) = Q(A)^G$ if A is a UFD and any rational homomorphism $G \to \mathbb{G}_m$ is trivial.

6.11 Let G be an algebraic group acting regularly on an algebraic variety X and let H be a closed invariant subgroup of finite index. Suppose that a geometric quotient $Y = X/G$ exists. Show that geometric quotients X/H and $(X/H)/(G/H)$ exist and $X/G \cong (X/H)/(G/H)$. Is the same true without assuming that H is of finite index?

Chapter 7

Linearization of actions

7.1 Linearized line bundles

We have seen already in the proof of Lemma 3.5 that a rational action of an affine algebraic group G on an affine variety X can be "linearized". This means that we can G-equivariantly embed X in affine space $\mathbb{A}^n$ on which G acts via a linear representation. We proved this by considering the linear space spanned by the G-translates of generators of the algebra $\mathcal{O}(X)$. In this chapter we will do a similar construction for a normal projective algebraic variety. This will be our main tool for constructing quotients.

Recall that a regular map of a projective variety X to the projective space $\mathbb{P}^n$ is defined by choosing a line bundle L (or equivalently an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules, or a Cartier divisor D) and a set of its sections $s_0, \ldots, s_n$. The map is defined by sending $x \in X$ to the point $(s_0(x), \ldots, s_n(x)) \in \mathbb{P}^n$. This point is well-defined if for any $x \in X$ there is a section s_i such that $s_i(x) \neq 0$. Often we will be taking for $(s_0, \ldots, s_n)$ a basis of the space of sections $\Gamma(X, L)$ of L. The condition above says in this case that for any $x \in X$ there exists a section $s \in \Gamma(X, L)$ such that $s(x) \neq 0$. We say in this case that L is *base-point-free*. Let $\phi_L : X \to \mathbb{P}^n$ be a map defined by a base-point-free L. Of course, it depends on the choice of a basis; different choices define maps which are the same up to composition with a projective transformation of $\mathbb{P}^n$. If ϕ_L is a closed embedding, L is called *very ample*. If $L^N := L^{\otimes N}$ is very ample for some $N > 0$, then L is called *ample*.

We will often identify L with its total space $\mathbb{V}(L)$, which comes with a projection $\pi : \mathbb{V}(L) \to X$; locally $\mathbb{V}(L)$ is the product of X and the affine line

$\mathbb{A}^1$.

Definition. A G-linearization of L is an action $\bar{\sigma}\colon G \times L \to L$ such that

(i) the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\bar{\sigma}} & L \\ {\scriptstyle \mathrm{id}\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

is commutative,

(ii) the zero section of L is G-invariant.

A G-linearized line bundle (or a *line G-bundle*) over a G-variety X is a pair $(L, \bar{\sigma})$ consisting of a line bundle L over X and its linearization. A morphism of G-linearized line bundles is a G-equivariant morphism of line bundles.

It follows from the definition that for any $g \in G$ and any $x \in X$ the induced map of the fibres

$$\bar{\sigma}_x(g) : L_x \to L_{g\cdot x}$$

is a linear isomorphism.

We can view the set of such isomorphisms as an isomorphism of line bundles

$$\bar{\sigma}(g) : L \to g^*(L),$$

where we consider $g \in G$ as an automorphism $x \to g \cdot x$ of X. The axioms of the actions translate into the following 1-*cocycle condition*:

$$\bar{\sigma}(gg') = \bar{\sigma}(g') \circ g'^*(\bar{\sigma}(g)) : L \to g'^*(L) \to g'^*(g^*(L)) = (gg')^*(L). \qquad (7.1)$$

The collection of the isomorphisms $\bar{\sigma}(g)$ can also be viewed as an isomorphism of vector bundles

$$\Phi : \mathrm{p}_2^*(L) \to \sigma^*(L).$$

The cocycle condition (7.1) is translated into a condition on Φ which can be expressed by some commutative diagrams; this is left to the reader.

Using the definition of linearization by means of an isomorphism Φ it is easy to define an abelian group structure on the set of line G-bundles. If $\Phi : \mathrm{p}_2^*(L) \to \sigma^*(L)$ and $\Phi' : \mathrm{pr}_2^*(L') \to \sigma^*(L')$ are two line G-bundles, we define their tensor

product as the line bundle $L \otimes L'$ with the G-linearization given by the isomorphism:

$$\Phi \otimes \Phi' : \mathrm{p}_2^*(L \otimes L') = \mathrm{p}_2^*(L) \otimes \mathrm{pr}_2^*(L') \to \sigma^*(L \otimes L) = \sigma^*(L) \otimes \sigma^*(L').$$

Here we use the obvious property of the inverse image

$$f^*(L \otimes L') = f^*(L) \otimes f^*(L').$$

The zero element in this group is the trivial line bundle $X \times \mathbb{A}^1$ whose linearization is given by the product $\sigma \times \mathrm{id} : G \times X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$. This is called the *trivial linearization*. The inverse (L, Φ) is equal to (L^{-1}, Φ') with Φ' defined as the inverse of the transpose of Φ. One checks that this again satisfies the cocycle condition. The structure of an abelian group which we have just defined induces an abelian group structure on the set of isomorphism classes of line G-bundles. We denote this group by $\mathrm{Pic}^G(X)$. It comes with the natural homomorphism

$$\alpha : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$$

which is defined by forgetting the linearization.

Let us now describe the kernel of the homomorphism α. Observe first that if $f : L \to L'$ is an isomorphism of line bundles and $\Phi : \mathrm{pr}_2^*(L) \to \sigma^*(L)$ is a G-linearization of L, then we can define a G-linearization of L' by setting $\Phi' = \sigma^*(f)^{-1} \circ \Phi \circ \mathrm{pr}_2^*(f)$. Thus, if $\alpha((L, \bar{\sigma}))$ is isomorphic to the trivial bundle, we can replace it by an isomorphic line G-bundle to assume that L is trivial. This shows that $\mathrm{Ker}(\alpha)$ consists of isomorphism classes of linearizations on the trivial line bundle $L = X \times \mathbb{A}^1$.

We denote a point of $X \times \mathbb{A}^1$ by (x, t). For any $g \in G$,

$$\bar{\sigma}(g)(x, t) = (g \cdot x, \Psi(g, x)t),$$

where $\Psi(g, x) \in k^*$. The function $\Psi : (g, x) \mapsto \Psi(g, x)$ must be a regular function on $G \times X$ which is nowhere vanishing. In other words, $\Psi \in \mathcal{O}(G \times X)^*$. The axioms of the action give us that

$$\Psi(gg', x) = \Psi(g, g' \cdot x)\Psi(g', x). \tag{7.2}$$

Let us see when two functions Ψ, Ψ' define isomorphic linearizations. Let $a : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ be an automorphism of the trivial bundle. It is defined by

a formula $(x,t) \mapsto (x, \phi(x)t)$, where $\phi \in \mathcal{O}(X)^*$. It commutes with the actions defined by Ψ and Ψ' if and only if

$$\phi(g \cdot x)\Psi(g,x) = \phi(x)\Psi'(g,x).$$

Or, equivalently, for any $g \in G$,

$$\Psi'(g,x) = \Psi(g,x)\phi(g \cdot x)/\phi(x).$$

Let $Z^1_{\text{alg}}(G, \mathcal{O}(X)^*)$ denote the group of functions Ψ satisfying (7.2) considered as a subgroup of the group $\mathcal{O}(G \times X)^*$ and let $B^1_{\text{alg}}(G, \mathcal{O}(X)^*)$ be its subgroup consisting of functions of the form $g^*(\phi)/\phi$ for some $\phi \in \mathcal{O}(X)^*$. It follows from the definition of the group structure on $\text{Pic}^G(X)$ that the product in $Z^1_{\text{alg}}(G, \mathcal{O}(X)^*)$ corresponds to the tensor product of linearized line G-bundles. So the above discussion proves the following.

Theorem 7.1. *The kernel of the forgetful homomorphism* $\alpha : \text{Pic}^G(X) \to \text{Pic}(X)$ *is isomorphic to the group*

$$H^1_{\text{alg}}(G, \mathcal{O}(X)^*) := Z^1_{\text{alg}}(G, \mathcal{O}(X)^*)/B^1_{\text{alg}}(G, \mathcal{O}(X)^*).$$

Note the special case when for any integral k-algebra K

$$(\mathcal{O}(X) \otimes_k K)^* = K^* \otimes 1.$$

This happens, for example, when X is affine space, or when X is connected and proper over k. Then

$$\mathcal{O}(G \times X)^* = \text{pr}_1^* (\mathcal{O}(G)^*)$$

and (7.2) gives that

$$Z^1_{\text{alg}}(G, \mathcal{O}(X)^*) \cong \text{Hom}_{\text{alg}}(G, \mathbb{G}_m) := \mathcal{X}(G),$$

the subscript indicating that we are considering rational homomorphisms of algebraic group. The latter group is called the *group of rational characters* of G. We studied this group when G was a torus. Also we have $g^*(\phi) = \phi$ and hence $B^1_{\text{alg}}(G, \mathcal{O}(X)^*) = 0$. Thus we obtain

Corollary 7.1. *Assume* $\mathcal{O}(G \times X)^* = \text{pr}_1^* (\mathcal{O}(G)^*)$. *Then*

$$\text{Ker}(\alpha) \cong \mathcal{X}(G).$$

Remark 7.1. According to a theorem of Rosenlicht ([94]), for any two irreducible algebraic varieties X and Y over an algebraically closed field k, the natural homomorphism

$$\mathcal{O}(X)^* \otimes \mathcal{O}(Y)^* \to \mathcal{O}(X \times Y)^*$$

is surjective. Let us give a sketch of the proof. First we use that for any irreducible algebraic variety the group $\mathcal{O}(X)^*/k^*$ is finitely generated. (This is not difficult to prove by reducing to the case of a normal variety and then finding a complete normal variety $\bar{X}$ containing X such that $D = \bar{X} \setminus X$ is a divisor. Then for any $f \in \mathcal{O}(X)^*$ the divisor of f is contained in the support of D and hence is equal to a linear combination of irreducible components of D. This defines an injective homomorphism from the group $\mathcal{O}(X)^*/k^*$ to a finitely generated abelian group.) Now assume we have an invertible function $\phi(x, y)$ on $X \times Y$. For a fixed $x \in X$ we have a function $\phi_x(y) = \phi(x, y) \in \mathcal{O}(Y)^*$. Since $\mathcal{O}(Y)^*/k^*$ is a finitely generated group, the map $X \to \mathcal{O}(Y)^*/k^*, x \mapsto \phi_x(y)$ modulo k^* must be constant. Of course to justify this we have to show that this map is given by an algebraic function; this can be done. So assuming this, we obtain that $\phi(x, y)$ is equal to a function $\psi(y)$ up to a multiplicative factor $c(x)$ depending on x. So $\phi(x, y) = c(x)\psi(y)$ as asserted.

7.2 The existence of linearization

To find conditions for the existence of a G-linearization of a line bundle we have to study the image of the forgetful homomorphism α. This consists of isomorphism classes of line bundles on X which admit some G-linearization. We start with the following lemma.

Lemma 7.1. *Let G be a connected affine algebraic group, and let X be an algebraic G-variety. A line bundle L over X admits a G-linearization if and only if there exists an isomorphism of line bundles $\Phi : \mathrm{p}_2^*(L) \to \sigma^*(L)$.*

Proof. We already know that this condition is necessary, so we show that it is sufficient. Assume that such an isomorphism exists. The problem is that it may not satisfy the cocycle condition (7.1). Let us interpret Φ as a collection of isomorphisms $\Phi_g : L \to g^*(L)$. When $g = e$, the unity element, we get an automorphism $\Phi_e : L \to L$. It is given by a function $\phi \in \mathcal{O}(X)^*$. Composing all Φ_g with Φ_e^{-1}, we may assume that $\Phi_e = \mathrm{id}_L$. Now the isomorphisms $\Phi_{gg'}$ and $g'^*(\Phi_g) \circ \Phi_{g'}$ differ by an automorphism of L. Denote it by $F(g, g')$ so that we have

$$\Phi_{gg'} \circ F(g, g') = g'^*(\Phi_g) \circ \Phi_{g'}.$$

The cocycle condition means that $F(g,g') \equiv \mathrm{id}_L$. So far we have only that $F(e,g) = F(g,e) = \mathrm{id}_L$ for any $g \in G$. Let us identify the automorphism $F(g,g')$ with an invertible function on $G \times G \times X$. By Rosenlicht's Theorem which we cited in Remark 7.1, we can write $F(g,g')(x) = F_1(g)F_2(g')F_3(x)$. Since $F(e,g',x) \equiv 1$ and $F(g,e,x) \equiv 1$, the functions $F_2(g)F_3(x)$ and $F_1(g)F_3(x)$ are constants. Thus $F_3(x)$ is constant and hence F_1 and F_2 are constants. This implies that $F \equiv 1$. This proves the assertion. □

Remark 7.2. The existence of an isomorphism $\Phi : \mathrm{p}_2^*(L) \to \sigma^*(L)$ means that L is a G-invariant line bundle. So the preceding lemma asserts that any G-invariant line bundle admits a G-linearization provided that G is a connected algebraic group. The assertion is not true if G is not connected. For example, assume that G is a finite group. The functions $F(g,g')$ which we considered in the preceding proof form a 2-cocycle of G with values in k^* (with trivial action of G in k^*). The obstruction for the existence of a G-linearization lies in the cohomology group $H^2(G,k^*)$. The latter group is called the group of *Schur multipliers* of G. It has been computed for many groups G and, of course, it is not trivial in general. If we denote the subgroup of G-invariant line bundles by $\mathrm{Pic}(X)^G$, then one has an exact sequence of abelian groups

$$0 \to \mathrm{Hom}(G,k^*) \to \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)^G \to H^2(G,k^*). \tag{7.3}$$

Lemma 7.2. *Assume that X is normal (for example, nonsingular) and G is a connected affine algebraic group. Let $x_0 \in X$. For any line bundle L on $G \times X$ we have*

$$L \cong \mathrm{pr}_1^*(L|G \times x_0) \otimes \mathrm{pr}_2^*(L|e \times X).$$

Proof. It is enough to show that $L \cong \mathrm{pr}_1^*(L_1) \otimes \mathrm{p}_2^*(L_2)$ for some $L_1 \in \mathrm{Pic}(G)$ and $L_2 \in \mathrm{Pic}(X)$; then it is immediately checked that $L_1 \cong L|G \times x_0$ and $L_2 \cong L|e \times X$. To do this we use the following fact about algebraic groups: G contains an open Zariski subset U isomorphic to $(\mathbb{A}^1 \setminus \{0\})^N$. For GL_n this follows from the fact that any matrix with nonzero pivots can be reduced to triangular form by elementary row transformations. We also use the fact that the homomorphism $\mathrm{pr}_2^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{A}^1 \setminus \{0\} \times X)$ is an isomorphism (see [46], Chapter 2, Proposition 6.6). These two facts imply that $L|U \times X \cong \mathrm{pr}_2^*(L_2)$ for some line bundle L_2 on X. Let D be a Cartier divisor on $G \times X$ representing L (i.e., $L \cong \mathcal{O}_{G \times X}(D)$). Then the preceding isomorphism implies that there exists a Cartier divisor D_2 on X such that $D' = D - \mathrm{pr}_2^*(D_2)|U \times X = 0$. For every irreducible component D_i' of D' its image in G is contained in the closed subset $Z = G \setminus U$. By the theorem

on the dimension of fibres, the fibres of $\mathrm{pr}_1 : D_i' \to Z$ must be of dimension equal to $\dim X$. This easily implies that $D_i' = \mathrm{pr}_1^*(D_i)$, where $D_i \subset Z$. Thus $D' = \mathrm{pr}_1^*(D_1)$ for some Weil (and hence Cartier because G is nonsingular) divisor on G. So we have the equality of Cartier divisors $D = \mathrm{pr}_1^*(D_1) + \mathrm{pr}_2^*(D_2)$. This translates into an isomorphism of line bundles $L \cong \mathrm{pr}_1^*(\mathcal{O}_G(D_1)) \otimes \mathrm{pr}_2^*(\mathcal{O}_X(D_2))$. □

Define now a homomorphism $\delta : \mathrm{Pic}(X) \to \mathrm{Pic}(G)$ by

$$\delta(L) = (\mathrm{pr}_2^*(L) \otimes \sigma^*(L^{-1}))|G \times x_0,$$

where x_0 is a chosen point in X. Suppose $\delta(L)$ is trivial. By the preceding lemma applied to $M = \mathrm{pr}_2^*(L) \otimes \sigma^*(L^{-1})$ we obtain that $M = \mathrm{pr}_2^*(M|e \times X)$. But the restriction of σ and pr_2 to $e \times X$ are equal. This implies that M is trivial, hence there exists an isomorphism $\Phi : \mathrm{pr}_2^*(L) \to \sigma^*(L)$. By Lemma 7.1, L admits a G-linearization. This proves

Theorem 7.2. *Let G be a connected affine algebraic group acting on a normal variety X. Then the following sequence of groups is exact*

$$0 \to \mathrm{Ker}(\alpha) \to \mathrm{Pic}^G(X) \xrightarrow{\alpha} \mathrm{Pic}(X) \xrightarrow{\delta} \mathrm{Pic}(G).$$

Corollary 7.2. *Under the assumption of the theorem, the image of $\mathrm{Pic}^G(X)$ in $\mathrm{Pic}(X)$ is of finite index. In particular, for any line bundle L on X there exists a number n such that $L^{\otimes n}$ admits a G-linearization.*

Proof. Use the fact that for any affine algebraic k-group G the Picard group $\mathrm{Pic}(G)$ is finite (see [65], p.74). □

Remark 7.3. The assertion that $\mathrm{Pic}(G)$ is finite can be checked directly for many groups. For example, the group is trivial for $G = \mathrm{GL}_n, \mathbb{G}_m^n, \mathbb{G}_a$ since these groups are open subsets of affine space. To compute $\mathrm{Pic}(G)$ for $G = \mathrm{PGL}_n, \mathrm{SL}_n$, we use the following facts. Let V be an irreducible hypersurface of degree d in $\mathbb{P}^N$. Then

$$\mathrm{Pic}(\mathbb{P}^N \setminus V) \cong \mathbb{Z}/d\mathbb{Z}. \tag{7.4}$$

This isomorphism is defined by restricting a sheaf to an open subset. Another fact, which is not trivial, is that

$$\mathrm{Pic}(V) = \mathbb{Z}h, \tag{7.5}$$

where h is the class of a hyperplane section of V, provided $N \geq 4$. This is called the *Lefschetz Theorem* on hyperplane sections (see [40], p. 169).

Now notice that $G = \mathrm{PGL}_n$ is isomorphic to $\mathbb{P}^{n^2-1} \setminus V$, where V is given by the determinant equation $\det(x_{ij}) = 0$. This gives

$$\mathrm{Pic}(\mathrm{PGL}_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

On the other hand, SL_n is isomorphic to the complement of a hyperplane section $x_{00} = 0$ of the hypersurface

$$\det((x_{ij})_{1\le i,j\le n}) - x_{00}^n = 0$$

in $\mathbb{P}^{n^2}$. So when $n \ge 2$ we can apply (7.4) to obtain

$$\mathrm{Pic}(\mathrm{SL}_n) \cong 0.$$

There is a notion of a simply connected semi-simple algebraic group (which makes sense over an arbitrary algebraically closed field). For all such groups $\mathrm{Pic}(G)$ is trivial. Any G is isomorphic to a quotient $\tilde{G}/A$, where $\tilde{G}$ is simply connected and A is a finite abelian group whose dual abelian group is isomorphic to $\mathrm{Pic}(G)$. For example, $\tilde{G} = \mathrm{SL}_n$ for $G = \mathrm{PGL}_n$. For simple algebraic groups $\mathrm{Pic}(G)$ is a subgroup of the abelian group $A(\mathcal{D})$ defined by the Cartan matrix of the root system of the Lie algebra of G. Here are the values of $A(\mathcal{D})$ for different types of simple Lie algebras:

A_n	B_n	C_n	D_{2k}	D_{2k+1}	F_4	G_2	E_6	E_7	E_8
$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	1	1	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1

We refer to [87] for a description of the Picard group of any homogeneous space G/H.

7.3 Linearization of an action

Now we are ready to prove that any algebraic action on a normal quasi-projective variety can be linearized. Let L be a G-linearized line bundle, let $V = \Gamma(X, L)$ be its space of sections, and let G be an affine algebraic group. The group G acts naturally and linearly on V by the formula

$$\rho(g)(s)(x) = \bar{\sigma}(g, s(\sigma(g^{-1}, x))),$$

or, in simplified notation,

$$(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x). \tag{7.6}$$

We know that any finite-dimensional subspace W' of V is contained in a G-invariant finite-dimensional subspace W generated by the translates of a basis of W'. Thus we obtain a linear representation

$$\rho : G \to \mathrm{GL}(W).$$

Assume that the linear system W is *base-point-free* (i.e., for any $x \in X$ there exists $s \in W$ such that $s(x) \neq 0$). Then W defines a regular map $\phi_W : X \to \mathbb{P}(W^*)$ by the formula

$$\phi_W(x) = \{s \in W : s(x) = 0\}.$$

Here we identify a point in $\mathbb{P}(W^*)$ with a hyperplane in W. Note that although "$s(x)$" does not make sense (since it depends on a local trivialization of L), the equality $s(x) = 0$ is well-defined. The representation (7.6) in W defines a representation in W^* and the induced projective representation in $\mathbb{P}(W^*)$. It is given by the formula

$$g \cdot H = g^{-1}(H),$$

where H is a hyperplane in W. Now

$$\begin{aligned} \phi_W(g \cdot x) &= \{s \in W : s(g \cdot x) = 0\} \\ = \{s \in W : g^{-1}s(g \cdot x) = 0\} &= \{s \in W : (g^{-1} \cdot s)(x) = 0\} \\ = g^{-1}(\phi_W(x)) &= g \cdot \phi_W(x). \end{aligned}$$

This shows that the map ϕ_W is G-equivariant.

Choosing a basis $(s_0, \ldots, s_n)$ in W we obtain a G-equivariant rational map

$$f : X \to \mathbb{P}^n, \quad x \mapsto (s_0(x), \ldots, s_n(x)).$$

If the rational map defined by a basis of W' is an embedding, then this map is an embedding too. Now let $i : X \hookrightarrow \mathbb{P}^N$ be an embedding of X as a locally closed subvariety of projective space. We take $L = i^*(\mathcal{O}_{\mathbb{P}^N}(1))$. When n is large enough, $L^{\otimes n} = i^*(\mathcal{O}_{\mathbb{P}^N}(n))$ admits a G-linearization. Let $W' \subset \Gamma(X, L^{\otimes n})$ be the image of $\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n))$ under the canonical restriction map $\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to \Gamma(X, L^{\otimes n})$. Obviously, W' is a finite-dimensional base-point-free linear system. It defines an embedding of X into projective space which is the composition of i and a Veronese map $v_n : \mathbb{P}^N \to \mathbb{P}^{\binom{N+n}{n}-1}$ (obtained from the Veronese map $v_d : \mathbb{P}(k^{N+1}) \to \mathbb{P}(\mathrm{Pol}_n(k^{N+1})$ by choosing bases). Replacing W' with a G-invariant linear system W as above, we obtain a linearization of the action of G on X.

Theorem 7.3. *Let X be a quasi-projective normal algebraic variety, acted on by an irreducible algebraic group G. Then there exists a G-equivariant embedding $X \hookrightarrow \mathbb{P}^n$, where G acts on $\mathbb{P}^n$ via its linear representation $G \to \mathrm{GL}_{n+1}$.*

Example 7.1. Let $G = \mathrm{PGL}_{n+1}$ act on $X = \mathbb{P}^n$ in the natural way. Let us see that the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is not G-linearizable but $\mathcal{O}_{\mathbb{P}^n}(n+1) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes(n+1)}$ is. We view G as an open subset of the projective space $\mathbb{P}^N (N = n^2 + 2n = \dim G)$ whose complement is the determinant hypersurface Δ given by the equation $\det((T_{ij})) = 0$. The action $\sigma\colon G \times X \to X$ is the restriction to $G \times X$ of the rational map $\sigma'\colon \mathbb{P}^N \times \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by the formula

$$\sigma'((a_{ij}), (x_0, \ldots, x_n)) = \Big(\sum_{j=0}^{n} a_{1j}x_j, \ldots, \sum_{j=0}^{n} a_{nj}x_j\Big).$$

Note that this map is not defined at any point (A, x) such that $\det(A) = 0, A \cdot x = 0$. The restriction of the projection $\mathbb{P}^N \times X \to \mathbb{P}^N$ to the set Z of such points is a birational map onto the determinant hypersurface (it is an isomorphism over the subset of matrices of corank equal to 1). Since Z is of codimension ≥ 2 in $\mathbb{P}^N \times \mathbb{P}^n$ the line bundle $\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1))$ is the restriction of a line bundle on $\mathbb{P}^N \times \mathbb{P}^n$. The formula for the action shows that this bundle must be $\mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^N}(1)) \otimes \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Thus $\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1))$ restricted to $(\mathbb{P}^N \setminus \Delta) \times \{x_0\}$ is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to $\mathbb{P}^N \setminus Z$. If $\mathcal{O}_{\mathbb{P}^n}(1)$ admits a linearization, we have $\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^n}(1))$, and hence the latter line bundle must be trivial. However, by (7.4), it is a generator of the group $\mathrm{Pic}(\mathbb{P}^N \setminus \Delta) \cong \mathbb{Z}/(n+1)\mathbb{Z}$.

Bibliographical notes

The existence of a linearization of some power of a line bundle on a normal complete algebraic variety was first proven in [74] by using the theory of Picard varieties for complete normal varieties. Our proof, which is borrowed from [65], does not use the theory of Picard varieties and applies to any normal quasi-projective varieties. One can also consider vector G-bundles of arbitrary rank (see for example [100]); however, no generalization of Corollary 7.2 to this case is known to me.

Exercises

7.1 Let L be a line bundle over a connected affine algebraic group. Show that the complement L^* of the zero section of L has the structure of an algebraic group such that the projection map $\pi : L^* \to G$ is a homomorphism of groups with kernel isomorphic to $\mathbb{G}_m$.

7.2 Let G be a connected affine algebraic group. Show that $H^1_{\text{alg}}(G, \mathcal{O}(X)^*)$ is a homomorphic image of the group $\mathcal{X}(G)$. In particular it is trivial if X is connected and complete.

7.3 Use Rosenlicht's Theorem from Remark 7.1 to show that any invertible regular function $f \in \mathcal{O}(G)^*$ on a connected affine algebraic group G with value 1 at the unity $e \in G$ defines a rational character of G.

7.4 Let X be a nonsingular algebraic variety and let G be its finite group of automorphisms. Show that the group $\text{Pic}^G(X)$ is isomorphic to the group of G-invariant Weil divisors modulo linear equivalence defined by G-invariant rational functions. [Hint: Use Hilbert's Theorem 90 which asserts that $H^1(G, k(X)^*) = 0$.]

7.5 Let $\mathbb{G}_m$ act on an affine algebraic variety X defining the corresponding grading of $\mathcal{O}(X)$. Let M be a projective module of rank 1 over $\mathcal{O}(X)$ and let L be the associated line bundle on X. Show that there is a natural bijective correspondence between G-linearizations of L and structures of $\mathcal{O}(X)$-graded modules on M.

7.6 Show that any line bundle on a normal irreducible variety X on which SL_n acts admits a unique SL_n-linearization.

7.7 Let $f : X \to \mathbb{P}(V)$ be a G-equivariant map, where G acts on $\mathbb{P}(V)$ via its linear representation. Show that $L = f^*(\mathcal{O}_{\mathbb{P}(V)}(1))$ admits a G-linearization and the map f is the map given by the line bundle L.

7.8 Show that the total space of the line bundle $L = \mathcal{O}_{\mathbb{P}^n}(1)$ is isomorphic to the complement of a point in $\mathbb{P}^{n+1}$. Describe the unique SL_{n+1}-linearization of L in terms of an action of the group SL_{n+1} on the total space.

Chapter 8

Stability

8.1 Stable points

From now on we will assume that G is a reductive algebraic group acting on an irreducible algebraic variety X. In this chapter we will explain a general construction of quotients due to D. Mumford. The idea is to cover X by open affine G-invariant sets U_i and then to construct the categorical quotient $X/\!\!/G$ by gluing together the quotients $U_i/\!\!/G$. The latter quotients are defined by Nagata's Theorem. Unfortunately, such a cover does not exist in general. Instead we find such a cover of some open subset of X. So we can define only a "partial" quotient $U/\!\!/G$. The construction of U will depend on a parameter, a choice of a G-linearized line bundle L.

Definition. Let L be a G-linearized line bundle on X and $x \in X$;

(i) x is called *semi-stable* (with respect to L) if there exists $m > 0$ and $s \in \Gamma(X, L^m)^G$ such that $X_s = \{y \in X : s(y) \neq 0\}$ is affine and contains x.

(ii) x is called *stable* (with respect to L) if there exists s as in (i) and additionally G_x is finite and all orbits of G in X_s are closed.

(iii) x is called *unstable* (with respect to L) if it is not semi-stable.

We shall denote the set of semi-stable (respectively stable, unstable) points by

$$X^{\mathrm{ss}}(L), \quad X^{\mathrm{s}}(L), \quad X^{\mathrm{us}}(L).$$

Remark 8.1. 1. Obviously $X^{ss}(L)$ and $X^{s}(L)$ are open G-invariant subsets (but could be empty).

2. If L is ample and X is projective, the sets X_s are always affine, so this condition in the definition of semi-stable points can be dropped. In fact, for any $n > 0$, $X_{s^n} = X_s$ so we may assume that L is very ample. Let $f : X \to \mathbb{P}_k^N$ be a closed embedding defined by some complete linear system associated to L. Then X_s is equal to the inverse image of an affine open subset in $\mathbb{P}_k^N$ which is the complement to some hyperplane. Since a closed subset of an affine set is affine we obtain the assertion.

3. The restriction of L to $X^{ss}(L)$ is ample. This is a consequence of the following criterion of ampleness: L is ample on a variety X if and only if there exists an affine open cover of X formed by the sets X_s, where s is a global section of some tensor power of L. For the proof we refer to [46], p. 155.

4. The definitions of the sets $X^{ss}(L), X^{s}(L), X^{us}(L)$ do not change if we replace L by a positive tensor power (as a G-linearized line bundle).

5. Assume L is ample. Let $x \in X^{ss}(L)$ be a point whose orbit $G \cdot x$ is closed and whose isotropy subgroup G_x is finite. I claim that $x \in X^{s}(L)$. In fact let $x \in X_s$ be as in the definition of semi-stable points. Then the set $Z = \{y \in X_s : \dim G_y > 0\}$ is closed in X_s and does not intersect $G \cdot x$. Since G is reductive, there exists a function $\phi \in \mathcal{O}(X_s)^G$ such that $\phi(G \cdot x) \neq 0, \phi(Z) = 0$. One can show that there exists some number $r > 0$ such that $\phi s^{\otimes r}$ extends to a section s' of some tensor power of L (see [46], Chapter 2, 5.14). Since X is irreducible, this section must be G-invariant. Thus $x \in X_{s'} \subset X_s$ and all points in $X_{s'}$ have zero-dimensional stabilizer. This implies that the orbits of all points in $X_{s'}$ are closed in $X_{s'}$. This checks that x is stable.

6. In [74] a stable point is called *properly stable* and in the definition of stability the finiteness of G_x is omitted.

Let us explain the definition of stability in more down-to-earth terms. Assume that L is very ample, and embed X equivariantly in $\mathbb{P}(V)$. We have a G-equivariant isomorphism of vector spaces

$$\Gamma(X, L^m) \cong \mathrm{Pol}_m(V)/I_m,$$

where I_m is the subspace of $\mathrm{Pol}_m(V)$ which consists of polynomials vanishing on X. Passing to invariants, we obtain

$$\Gamma(X, L^m)^G \cong (\mathrm{Pol}_m(V)/I_m)^G.$$

Let x^* denote a point in V such that $kx^* = x \in \mathbb{P}(V)$. Every $s \in \Gamma(X, L^m)^G$ can be represented by a polynomial $F_s \in \mathrm{Pol}_m(V)$ which is G-invariant modulo

I_m. In particular, F_s is constant on the orbit of x^* for any point $x \in X$. Clearly $s(x) \neq 0$ if and only if F_s does not vanish on x^*. So the set of unstable points is equal to the image in $\mathbb{P}(V)$ of the set

$$\mathcal{N}(G;V) = \{v \in V : F(v) = 0, \forall F \in \bigoplus_{m>0} \mathrm{Pol}_m(V)^G\}.$$

This set is called the *null-cone* of the linear action of G in V. It is an affine variety given by a system of homogeneous equations (an *affine cone*). Let $v \in V$ and $\mathrm{O}(v)$ be its orbit in V. Suppose $0 \in \overline{\mathrm{O}(v)}$. Then for any G-invariant polynomial F we have $F(v) = F(\overline{\mathrm{O}(v)}) = F(0) = 0$. Thus the corresponding point $x = kv$ in X is unstable. Conversely, if x is unstable, $0 \in \overline{\mathrm{O}(v)}$. In fact, otherwise we can apply Lemma 6.1 and find an invariant polynomial P such that $P(v) \neq 0$ but $P(0) = 0$. If we write P as a sum of homogeneous polynomials P_m of positive degree, we find some P_m which does not vanish at v. Then x is semi-stable. This interpretation of stability goes back to the original work of D. Hilbert ([47]).

8.2 The existence of a quotient

Let us show that the open subset of semi-stable (respectively stable) points admits a categorical (respectively geometric) quotient.

First we have to recall the definition of the gluing construction of algebraic varieties. Let $\{X_i\}_{i\in I}$ be a finite set of affine algebraic varieties. The *gluing data* is a choice of an open affine subset $U_{ij} \subset X_i$ for each $j \in I$, and an isomorphism $\phi_{ji} : U_{ij} \to U_{ji}$ for each pair $(i,j) \in I \times I$. It is required that

(i) $U_{ii} = X_i$, and ϕ_{ii} is the identity for each $i \in I$,

(ii) for any $i,j,k \in I$, $\phi_{ji}(U_{ij} \cap U_{ik}) \subset U_{jk}$ and

$$(\phi_{kj} \circ \phi_{ji})|_{U_{ij}\cap U_{ik}} = \phi_{ki}|_{U_{ij}\cap U_{ik}}.$$

Let R be an equivalence relation on the set $\bigsqcup_{i\in I} X_i$ defined by $x \sim y$ if and only if there exists a pair $(i,j) \in I \times I$ such that $x \in U_{ij}, y \in U_{ij}$ and $y = \phi_{ij}(x)$. The assumptions (i) and (ii) show that it is indeed an equivalence relation. Let X be the corresponding factor set and let $p : \bigsqcup_{i\in I} X_i \to X$ be the canonical projection. We equip X with the topology for which a subset V is open if and only if $p^{-1}(V)$ is open in the Zariski topology. The restriction p_i of p to X_i defines a homeomorphism of X_i with an open subset V_i of X so that $X = \bigcup_{i\in I} V_i$ and $p_i(U_{ij}) = V_i \cap V_j$. We also introduce the notion of a regular function on an

open subset $V \subset X$. By definition, this is a collection of regular functions f_i on $p_i^{-1}(V) \subset X_i$ such that $(f_j|_{U_{ji}}) \circ \phi_{ji} = f_i|_{U_{ij}}$ for any $i, j \in I$. Let $\mathcal{O}_X(V)$ be the k-algebra of regular functions on V. The assignment $V \to \mathcal{O}_X(V)$ is a sheaf of k-algebras, called the *structure sheaf* of X. The pair $(X, \mathcal{O}_X)$ is an example of a *ringed space*, i.e., a topological space equipped with a sheaf of rings. Ringed spaces form a category. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ such that for any open subset $V \subset Y$ and $\phi \in \mathcal{O}_Y(V)$, the composition $\phi \circ f \in \mathcal{O}_X(f^{-1}(V))$. Each open subset V of X is equipped with the structure of a ringed space whose structure sheaf $\mathcal{O}_V$ is equal to the restriction of $\mathcal{O}_X$ to V. Each quasi-projective algebraic variety can be considered as a ringed space, the structure sheaf is the sheaf of regular functions. It follows from the definition that the ringed space $(X, \mathcal{O}_X)$ obtained by gluing of affine varieties is locally isomorphic to an affine variety, i.e., it admits an open cover by subsets which are isomorphic to affine varieties as ringed spaces; in the notation from above each open set V_i is isomorphic to X_i. Thus we are led to the notion of an *abstract algebraic variety* which is a ringed space locally isomorphic to an affine algebraic variety. One usually adds a separatedness property which ensures that the intersection of two open affine subsets is an affine variety. An abstract algebraic variety X is isomorphic to a quasi-projective algebraic variety if and only if there exists an ample line bundle L over X which is used to embed X into projective space. We leave it to the reader to define the notion of a line bundle over an abstract algebraic variety. A useful criterion of ampleness of a line bundle was given in Remark 8.1.3.

Theorem 8.1. *There exists a good categorical quotient*

$$\pi : X^{ss}(L) \to X^{ss}(L)/\!\!/G.$$

There is an open subset U in $X^{ss}(L)/\!\!/G$ such that $X^s(L) = \pi^{-1}(U)$ and the restriction of π to $X^s(L)$ is a geometric quotient of $X^s(L)$ by G. Moreover there exists an ample line bundle M on $X^{ss}(L)//G$ such that $\pi^*(M) = L^{\otimes n}$, restricted to $X^{ss}(L)$, for some $n \geq 0$. In particular, $X^{ss}(L)/\!\!/G$ is a quasi-projective variety.

Proof. Since any open subset of X is quasi-compact in the Zariski topology we can find a finite set $\{s_1, \ldots, s_r\}$ of invariant sections of some tensor power of L such that $X^{ss}(L)$ is covered by the sets X_{s_i}. Obviously we may assume that all the s_i belong to $\Gamma(X, L^{\otimes N})^G$ for some sufficiently large N. Let $U_i = X_{s_i}, i = 1, \ldots, r$. For every U_i, we consider the ring $\mathcal{O}(U_i)^G$ of G–invariant regular functions and let $\pi_i : U_i \to Y_i := U_i/\!\!/G$ with $\mathcal{O}(Y_i) = \mathcal{O}(U_i)^G$ as constructed in

Nagata's Theorem. For each i, j we can consider s_i/s_j as a regular G-invariant function on U_j. Let $\phi_{ij} \in \mathcal{O}(Y_j)$ be the corresponding regular function on the quotient. Consider the open subset $D(\phi_{ij}) \subset Y_j$. Obviously

$$\pi_j^{-1}(D(\phi_{ij})) = \pi_i^{-1}(D(\phi_{ji})) = U_i \cap U_j.$$

This easily implies that both sets $D(\phi_{ij})$ and $D(\phi_{ji})$ are categorical quotients of $U_i \cap U_j$. By the uniqueness of categorical quotient there is an isomorphism $\alpha_{ij} : D(\phi_{ij}) \to D(\phi_{ji})$. It is easy to see that the set of isomorphisms $\{\alpha_{ij}\}$ satisfies the conditions of gluing. So we can glue together the quotients Y_i and the maps π_i to obtain a morphism $\pi : X^{ss}(L) \to Y$, where $Y = X^{ss}(L)/\!\!/G$. To show that Y is separated it is enough to observe that it admits an affine open cover by the sets Y_i which satisfies the following properties: $Y_i \cap Y_j \cong U_i \cap U_j/\!\!/G$ are affine and $\mathcal{O}(Y_i \cap Y_j)$ is generated by restrictions of functions from $\mathcal{O}(Y_i)$ and $\mathcal{O}(Y_j)$. The latter property follows from the fact that $\mathcal{O}(U_i \cap U_j)$ is generated by restrictions of functions from $\mathcal{O}(U_i)$ and $\mathcal{O}(U_j)$. In fact, the separatedness also follows from the assertion that Y is quasi-projective. So let us concentrate on proving the latter.

Note that the cover $\{U_i\}_{i=1,\dots,r}$ of $X^{ss}(L)$ is a trivializing cover for the line bundle L' obtained by restriction of L to $X^{ss}(L)$. In fact, by Remark 8.1.3, L' is ample; hence we may assume that some tensor power $L'^{\otimes t}$ is very ample. This implies that $L'^{\otimes t}$ is equal to the line bundle $f^*(\mathcal{O}_{\mathbb{P}^n}(1))$ for some embedding $f : X^{ss}(L) \to \mathbb{P}^n$. The section $s_i^{\otimes t}$ of $L'^{\otimes t}$ is equal to the section $f^*(h)$ where h is a section of $\mathcal{O}_{\mathbb{P}^n}(1)$. Thus the open subset U_i is equal to $f^{-1}(V_i)$ where V_i is an open subset of $\mathbb{P}^n$ isomorphic to affine space. This shows that L' restricted to U_i is equal to $(f|_{U_i})^*(\mathcal{O}_{\mathbb{P}^n}(1)|V_i)$. However, $\mathcal{O}_{\mathbb{P}^n}(1)|V$ is isomorphic to the trivial line bundle since any line bundle over affine space is isomorphic to the trivial bundle. By fixing some trivializing isomorphisms we can identify the functions $(s_i/s_j)|_{U_i \cap U_j}$ with the transition functions g_{ij} of L'. As we have shown before, $s_i/s_j = \pi^*(\phi_{ij})$ for some functions $\phi_{ij} \in \mathcal{O}(Y_j)$. We use the transition functions $h_{ij} = \phi_{ij}|Y_i \cap Y_j$ to define a line bundle M on Y. Obviously $\pi^*(M) \cong L'$. Let us show that M is ample. First we define some sections t_j by setting $t_j|_{Y_i} = \phi_{ij}$ for a fixed j and variable i. Since for any i_1, i_2

$$\phi_{i_2 j} = \phi_{i_1 j}\phi_{i_2 i_1}$$

the $t_j|_{Y_{i_1} \cap Y_{i_2}}$ differ by the transition function of M, hence t_j is in fact a section of M. Clearly $\pi^*(t_j) = s_j$ and $Y_{t_j} = Y_j$. As above, since all Y_j are affine, we obtain that M is ample. Since $\pi : X^{ss}(L) \to Y$ is obtained by gluing together good categorical quotients, the morphism π is a good categorical quotient.

It remains to show that the restriction of π to $X^s(L)$ is a geometric quotient. By definition $X^s(L)$ is covered by affine open G-invariant sets where G acts with closed orbits. Since π is a good categorical quotient, for any $x \in X^s(L)$ the fibre $\pi^{-1}(\pi(x))$ consists of one orbit. Thus $\pi|_{X^s(L)}$ is a good geometric quotient. □

In the case when L is ample and X is projective, the following construction of the categorical quotient $X^{ss}(L)/\!\!/G$ is equivalent to the previous one.

Proposition 8.1. *Assume that X is projective and L is ample. Let*

$$R = \bigoplus_{n \geq 0} \Gamma(X, L^{\otimes n}).$$

Then

$$X^{ss}(L)/\!\!/G \cong \mathrm{Proj}(R^G).$$

In particular, the quotient $X^{ss}(L)/\!\!/G$ is a projective variety.

Proof. First of all, we observe that by Nagata's Theorem the algebra R^G is finitely generated. It also has a natural grading, induced by the grading of R. Replacing L by $L^{\otimes d}$ we may assume that R^G is generated by elements $s_0, \ldots, s_n$ of degree 1. Let $Y = \mathrm{Projm}(R^G)$ be the projective subvariety of $\mathbb{P}^n$ corresponding to the homogeneous ideal I equal to the kernel of the homomorphism $k[T_0, \ldots, T_n] \to R^G, T_i \mapsto s_i$. (The reader should go back to Chapter 3 to recall the definition of $\mathrm{Projm}(A)$ for any finitely generated graded k-algebra A.) The elements s_i generate the ideal $\mathfrak{m} = R^G_+$ generated by homogeneous elements of positive degree. Thus the affine open sets $U_i = X_{s_i}$ cover $X^{ss}(L)$. On the other hand the open sets $Y_i = Y \cap \{T_i \neq 0\}$ form an open cover of Y with the property that $\mathcal{O}(Y_i) = \mathcal{O}(U_i)^G$. The maps $U_i \to Y_i$ define a morphism $X^{ss}(L) \to Y$ which coincides with the categorical quotient defined in the proof of the preceding theorem. □

Remark 8.2. If we assume that L is very ample, and embeds X in the projective space $\mathbb{P}(\Gamma(X, L)^*) = \mathbb{P}(V)$, then we can interpret the null-cone as follows. The sections s_i from the proof of the preceding proposition define a G-equivariant rational map $X \to \mathbb{P}^n, x \mapsto (s_0(x), \ldots, s_n(x))$. The closed subset of X where this map is not defined is exactly the closed subvariety of X equal to $X \cap \overline{\mathcal{N}(G; V)}$, where the bar denotes the image of the null-cone $\mathcal{N}(G; V)$ in $\mathbb{P}(V)$. So deleting this closed subset from X we obtain the set $X^{ss}(L)$ and the quotient map $X^{ss}(L) \to X^{ss}(L)/\!\!/G$.

Remark 8.3. Note that the morphism $X^{ss}(L) \to X^{ss}(L)/\!\!/G$ is affine, i.e., inverse image of an affine open set is affine. There is also the following converse of the preceding proposition. Let U be a G-invariant open subset of X such that the geometric quotient $\pi : U \to U/G$ exists and is an affine map. Assume U/G is quasi-projective. Then there exists a G-linearized line bundle L such that $U \subseteq X^s(L)$. For the proof we refer to [74], p. 41.

8.3 Examples

Example 8.1. Let $X = \mathbb{P}^n$ and $G = \mathrm{SL}_{n+1}$ acting on $\mathbb{P}^n$ naturally via its linear representation. We know that $L = \mathcal{O}_{\mathbb{P}^n}(1)$ admits a unique SL_{n+1} linearization (Exercise 7.7). We also know from Chapter 5 that $\mathrm{Pol}_m(k^{n+1})$ is an irreducible representation for G. Therefore, for any $m > 0$,

$$\Gamma(X, \mathcal{O}_{\mathbb{P}^n}(m))^G = \mathrm{Pol}_m(k^{n+1})^G = \{0\}.$$

This shows that $X^{ss}(L) = \emptyset$.

Example 8.2. Let $X = \mathbb{P}^n$, $G = \mathbb{G}_m$ with action defined by the formula

$$t \cdot (x_0, \dots, x_n) = (t^{q_0} x_0, \dots, t^{q_n} x_n).$$

Here $q_0, \dots, q_n$ are some integers. We assume that $q_0 \le q_1 \le \cdots \le q_n$. Since $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}\mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathcal{X}(\mathbb{G}_m) \cong \mathbb{Z}$ we have $\mathrm{Pic}^G(\mathbb{P}^n) \cong \mathbb{Z}^2$. A G-linearized bundle must be of the form $\mathcal{O}_{\mathbb{P}^n}(m)$; it defines a G-equivariant *Veronese embedding* $\mathbb{P}^n \to \mathbb{P}^{N(m)}$, where $N(m) = \dim k[T_0, \dots, T_n]_m - 1 = \binom{n+m}{m} - 1$. The group $\mathbb{G}_m$ acts on $\mathbb{P}^{N(m)}$ by the formula $t : x_{i_1 \dots i_m} \mapsto t^{q_{i_1} + \cdots + q_{i_m}} x_{i_1 \dots i_m}$, where $x_{i_1 \dots i_m}$ is the coordinate in the Veronese space corresponding to the monomial $x_{i_1} \dots x_{i_m}$, $i_1 \le \cdots \le i_m$. Now the linearization is given by a linear representation of $\mathbb{G}_m$ in the space $(k[T_0, \dots, T_n]_m)^*$ which lifts the action in the corresponding projective space. Obviously it is defined by the formula

$$t : x_{i_1 \dots i_m} \mapsto t^{-a} t^{q_{i_1} + \cdots + q_{i_m}} x_{i_1 \dots i_m}, \tag{8.1}$$

for some integer a. Thus the G-linearized bundles can be indexed by the pairs $(m, a) \in \mathbb{Z}^2$. Denote the corresponding line bundle by $L_{m,a}$. Raising $L_{m,a}$ to the rth power as a G-linearized bundle corresponds to replacing (m, a) with (rm, ra).

We know that $X^{ss}(L)$ does not change if we replace L by $L^{\otimes r}$. So we may assume that $L = L_{1,p/q}$, where by definition $H^0(\mathbb{P}^n, L_{1,p/q}^{\otimes N})^G$ is defined only for N

divisible by q and $H^0(\mathbb{P}^n, L_{1,p/q}^{\otimes N}) = H^0(\mathbb{P}^n, L_{N,Np/q})^G$. In other words we permit a to be a rational number in formula (8.1) and consider invariant polynomials of degree a multiple of the denominator of a. Here the invariance means that for any $t \in k^*$,

$$F(t^{-a+q_0}x_0, \ldots, t^{-a+q_n}x_n) = F(x_0, \ldots, x_n).$$

Assume now that $q_0 \leq 0$. It is obvious that $\Gamma(\mathbb{P}^n, L_{1,a}^{\otimes N})^G = 0$ for all $N > 0$ if $a \leq q_0$ or $a \geq q_n$. This implies that $X^{ss}(L_{1,a}) = \emptyset$ if $a \notin [q_0, q_n]$.

When $a = q_0$, we have

$$\bigoplus_{N=0}^{\infty} H^0(\mathbb{P}^n, L_{1,a}^{\otimes N})^G = k[T_0, \ldots, T_m],$$

if $q_0 = \cdots = q_m < q_{m+1}$. Hence

$$X^{ss}(L_{1,a}) = \mathbb{P}^n \setminus \{x_0 = \cdots = x_m = 0\}$$

and

$$X^{ss}(L_{1,a})/\!\!/G = \mathrm{Projm}(k[T_0, \ldots, T_m]) = \mathbb{P}^m.$$

In particular, if $q_1 > q_0$, the quotient is the point.

Next, we increase the parameter a. If $q_m < a \leq q_{m+1}$, we have further invariant polynomials. For example, if $a = s/d$, the monomial $T_0^{dq_{m+1}-s}T_{m+1}^{dq_0-s}$ belongs to $\bigoplus_{N=0}^{\infty} H^0(\mathbb{P}^n, L_{1,a}^{\otimes N})^G$. So the set $X^{ss}(L_{1,a})$ becomes larger and the categorical quotient changes. In fact one can show that the quotients do not change when a stays strictly between two different weights q_i and do change otherwise.

Example 8.3. Consider the special case of the previous example where $q_0 = 0$ and $q_1 = \cdots = q_n = 1$. The restriction of the action to $\mathbb{A}^n$ is given by the formula

$$t \cdot (z_1, \ldots, z_n) = (t \cdot z_1, \ldots, t \cdot z_n).$$

If we take $L = L_{1,a}$ for $a = 1/2$ we get

$$\bigoplus_{m=0}^{\infty} \Gamma(\mathbb{P}^n, L_{1,a}^{\otimes 2m})^G = k[T_0T_1, \ldots, T_0T_n].$$

This shows that $X^{us}(L) = V(T_0) \cup V(T_1, \ldots, T_n)$. In other words, the set of semi-stable points is equal to the complement of the hyperplane at infinity $T_0 = 0$ and the point $(1, 0, \ldots, 0)$. So it can be identified with $\mathbb{A}^n \setminus \{0\}$. The quotient

is of course $\mathbb{P}^{n-1}$. Since the group G acts on this set with trivial stabilizers, we obtain that all orbits are closed and the quotient is a good geometric quotient.

Similar conclusions can be made for any rational $a \in (0,1)$. If $a = 1$, we have

$$\bigoplus_{m=0}^{\infty} \Gamma(\mathbb{P}^n, L_{1,1}^{\otimes m})^G = k[T_1, \ldots, T_n].$$

Thus

$$X^{\mathrm{us}}(L) = \mathbb{P}^n \setminus V(T_1, \ldots, T_n) = \mathbb{P}^n \setminus \{(1, 0, \ldots, 0)\}.$$

The categorical quotient is the same $\mathbb{P}^n$ but the set of semi-stable points is different.

Example 8.4. Let $X = \mathbb{A}^n$ and $G = \mathbb{G}_m$. Every line bundle is isomorphic to the trivial bundle $L = X \times \mathbb{A}^1$. As we saw in Chapter 7, its G-linearization is defined by the formula

$$t \cdot (z, v) = (t \cdot z, \chi(t)v),$$

where $\chi : \mathbb{G}_m \to \mathbb{G}_m$ is a homomorphism of algebraic groups. It is easy to see that any such homomorphism is given by the formula $t \mapsto t^\alpha$ for some integer α. In fact $\chi^* : k[T, T^{-1}] \to k[T, T^{-1}]$ is defined by the image of T, and the condition that this map is a homomorphism implies that the image is a power of T. So let L_α denote the G-linearized line bundle which is trivial as a bundle and whose linearization is given by the formula

$$t \cdot (z, v) = (t \cdot z, t^\alpha v).$$

A section $s : X \to L_\alpha$ of L_α is given by the formula

$$s(z) = (z, F(z))$$

for some polynomial $F(Z) \in k[Z] = \mathcal{O}(\mathbb{A}^n)$. The group G acts on the space of sections by the formula $s \mapsto {}^t s$, where

$${}^t s(z) = (z, t^\alpha \cdot F(t^{-1} \cdot z)).$$

Thus $s \in \Gamma(X, L_\alpha^{\otimes m})^G$ if and only if

$$F(t \cdot z) = t^{m\alpha} \cdot F(z) \quad \text{for all } z \in k^n, t \in k^*.$$

When $\alpha = 0$, the constant polynomial 1 defines an invariant section of $L^{\otimes m}$ for any m. Thus $X^{\mathrm{ss}}(L_0) = X$ and

$$X /\!\!/ G = \mathrm{Spm}(\mathcal{O}(X)^G) = \mathrm{Spm}(k[Z_1, \ldots, Z_n]^{\mathbb{G}_m}).$$

Recall that a $\mathbb{G}_m$-action on an affine variety is equivalent to a $\mathbb{Z}$-grading of its ring of regular functions; the ring of invariants is the subring of elements of degree 0 (see Example 3.1). In our case $\mathcal{O}(X) \cong k[Z_1, \ldots, Z_n]$ but the variables Z_i are not necessarily homogeneous. If we can make a linear change of variables such that they are homogeneous, then the action is given by a formula

$$t \cdot (z_1, \ldots, z_n) = (t^{\alpha_1} z_1, \ldots, t^{\alpha_n} z_n).$$

In this case we say that the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ is *linearizable* . It is an open problem (a very difficult one) whether any action of $\mathbb{G}_m$ on affine space is linearizable. It is known to be true for $n \leq 3$.

Assume now that $\alpha > 0$. Since we know that the set of semi-stable points and the quotient do not change when we replace L by its tensor power, we may assume that $\alpha = 1$. Then

$$\bigoplus_{m=0}^{\infty} \Gamma(X, L_\alpha^{\otimes m})^{\mathbb{G}_m} = \bigoplus_{m=0}^{\infty} k[Z_1, \ldots, Z_n]_m := k[Z_1, \ldots, Z_n]_{\geq 0}.$$

The subring $k[Z_1, \ldots, Z_n]_{\geq 0}$ is a finitely generated algebra over $k[Z_1, \ldots, Z_n]_0$. Thus

$$\bigoplus_{m>0}^{\infty} \Gamma(X, L_\alpha^{\otimes m})^{\mathbb{G}_m} = k[Z_1, \ldots, Z_n]_{>0}$$

is a finitely generated ideal in $k[Z_1, \ldots, Z_n]_{\geq 0}$. Let $f_1, \ldots, f_m$ be its homogeneous generators. Then

$$\begin{aligned} X^{ss}(L_\alpha) &= D(f_1) \cup \cdots \cup D(f_m), \\ X^{ss}(L_\alpha)/\!\!/\mathbb{G}_m &= D_+(f_1) \cup \cdots \cup D_+(f_m), \end{aligned}$$

where $D_+(f_i) = \mathrm{Spm}(k[Z_1, \ldots, Z_n]_{(f_i)})$ (see Example 3.1).

Similar conclusion can be reached in the case $\alpha < 0$.

Example 8.5. A special case of the previous example is when $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by the formula

$$t \cdot (z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_n),$$

where $q_i > 0$. If $\alpha = 0$, we get $k[Z_1, \ldots, Z_n]_0 = k$ so the quotient is one point. If $\alpha < 0$, we get $k[Z_1, \ldots, Z_n]_{<0} = \{0\}$, so the set of semi-stable points is empty. Finally, if $\alpha > 0$, we get

$$X^{ss} = D(Z_1) \cup \cdots \cup D(Z_n) = \mathbb{A}^n \setminus \{0\},$$

and the construction of the categorical quotient coincides with the construction of the weighted projective space $\mathbb{P}(q_1, \ldots, q_n)$ (see Example 3.1). So we see two different ways to define $\mathbb{P}^n$: as a quotient of $\mathbb{P}^{n+1}$ and as a quotient of $\mathbb{A}^{n+1}$.

Example 8.6. Let G be again $\mathbb{G}_m$ and $X = \mathbb{A}^4$ with the action given by the formula

$$t \cdot (z_1, z_2, z_3, z_4) = (tz_1, tz_2, t^{-1}z_3, t^{-1}z_4).$$

As in the previous example, each G-linearized line bundle is isomorphic to the trivial line bundle with the G-linearization defined by an integer α. We have

$$\Gamma(X, L_\alpha^{\otimes r})^G = k[Z]_{r\alpha}.$$

However, this time the grading is weighted; the weights are $(1, 1, -1, -1)$.

Assume $\alpha = 0$. Then for any $r > 0, 1 \in \Gamma(X, L_0^{\otimes r})^G = \Gamma(X, L_0)^G$. Hence $X = X^{ss}(L)$, and

$$\mathcal{O}(X)^G = k[Z]_0 = k[Z_1Z_3, Z_1Z_4, Z_2Z_3, Z_2Z_4] \subset k[Z].$$

We have a surjection $k[T_1, T_2, T_3, T_4] \to \mathcal{O}(X)^G$, defined by $T_1 \mapsto Z_1Z_3, T_2 \mapsto Z_1Z_4$, $T_3 \mapsto Z_2Z_3$, $T_4 \mapsto Z_2Z_4$. This shows that

$$\mathcal{O}(X)^G \cong k[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3).$$

Thus $X^{ss}(L)/\!\!/\mathbb{G}_m$ is isomorphic to the closed subvariety Y_0 of $\mathbb{A}^4$ given by the equation

$$T_1T_4 - T_2T_3 = 0.$$

This is a quadric cone. It has one singular point at the origin.

Assume $\alpha > 0$. Again, without loss of generality we may take $\alpha = 1$. It is easy to see that

$$\bigoplus_{r>0} k[Z]_r = k[Z]_{>0} = Z_1k[Z]_{\geq 0} + Z_2k[Z]_{\geq 0}.$$

Thus

$$X^{ss}(L_1) = \mathbb{A}^4 \setminus V(Z_1, Z_2).$$

This set is covered by $U_1 = D(Z_1)$ and $U_2 = D(Z_2)$. We have

$$\begin{aligned} \mathcal{O}(U_1)^G &= k[Z]_{(Z_1)} = k[Z]_0[Z_2/Z_1], \\ \mathcal{O}(U_2)^G &= k[Z]_{(Z_2)} = k[Z]_0[Z_1/Z_2]. \end{aligned}$$

We claim that $X^{ss}(L_1)/G$ is isomorphic to a closed subvariety Y' of $\mathbb{A}^4 \times \mathbb{P}^1_k$ given by the equations

$$T_1Z_2 - T_3Z_1 = 0, \quad T_2Z_2 - T_4Z_1 = 0, \quad T_1T_4 - T_2T_3 = 0.$$

Here we use (Z_1, Z_2) for homogeneous coordinates in $\mathbb{P}^1$. In fact, this variety is covered by the two affine open sets Y'_i given by $Z_i \neq 0, i = 1, 2$. It is easy to see that $\mathcal{O}(Y'_i) \cong \mathcal{O}(U_i)^G$. We also verify that these two sets are glued together as they should be according to our construction of the categorical quotient. Thus we obtain an isomorphism $Y' \cong Y_+ := X^{ss}(L_1)/\!\!/\mathbb{G}_m$. In fact, we have $X^{ss}(L_1) = X^s(L_1)$ so that Y_+ is a geometric quotient. Note that we have a canonical morphism

$$f_+ : Y_+ \to Y_0$$

which is given by the inclusion of the rings $k[Z]_0 \subset \mathcal{O}(U_i)^G$. Geometrically it is induced by the projection $\mathbb{A}^4 \times \mathbb{P}^1 \to \mathbb{A}^4$. Over the open subset $Y_0 \setminus \{0\}$ this morphism is an isomorphism. In fact, $Y_0 \setminus \{0\}$ is covered by the open subsets $U_i = Y_0 \cap D(T_i), i = 1, \ldots, 4$. The inverse image $\overline{U_1} = f_+^{-1}(U_1)$ is contained in the open subset where $Z_1 \neq 0$. Since $Z_2/Z_1 = T_3/T_1$ we see that f_+ induces an isomorphism $\mathcal{O}(U_1) \to \mathcal{O}(\overline{U_1})$. Similarly we treat the other pieces U_i. Over the origin, the fibre of f_+ is isomorphic to $\mathbb{P}^1$. Also, we immediately check that Y_+ is a nonsingular variety. Thus $f_+ : Y_+ \to Y_0$ is a resolution of singularities of Y_0. It is called a *small resolution* because the exceptional set is of codimension > 1. The reader familiar with the notion of the blowing up will recognize Y_+ as the variety obtained by blowing up the closed subvariety of Y_0 defined by the equations $T_1 = T_3 = 0$.

Assume $\alpha < 0$. Similar arguments show that $Y_- = X^s(L_{-1})/\mathbb{G}_m$ is isomorphic to the closed subvariety of $\mathbb{A}^4 \times \mathbb{P}^1_k$ given by the equations

$$T_1Z_4 - T_2Z_3 = 0, T_3Z_4 - T_4Z_3 = 0, T_1T_4 - T_2T_3 = 0.$$

We have a morphism

$$f_- : Y_- \to Y_0$$

which is an isomorphism over $Y_0 \setminus \{0\}$ and whose fibre over $\{0\}$ is isomorphic to $\mathbb{P}^1$. The diagram

$$\begin{array}{ccc} Y_+ & & Y_- \\ {\scriptstyle f_+}\searrow & & \swarrow{\scriptstyle f_-} \\ & Y & \end{array}$$

represents a type of birational transformations between algebraic varieties which nowadays is called a "■ip". Note that Y_+ is not isomorphic to Y_-, but they are isomorphic outside the fibres $f_\pm^{-1}(0) \cong \mathbb{P}^1$.

Bibliographical notes

The theory of stable points with respect to an algebraic action was developed in [74]. There is nothing original in our exposition. The examples given in the chapter show the dependence of the sets of stable points on the choice of linearization of the action. Although this fact was implicitly acknowledged in [74], the serious study of this dependence began only recently; see [23], [116] and the references there. One of the main results of the theory developed in these papers is the finiteness of the set of open subsets which can be realized as the set of semi-stable points for some linearization.

Exercises

8.1 Let X be a homogeneous space with respect to an action of an affine algebraic group G. Assume X is not affine. Show that for any $L \in \mathrm{Pic}^G(X)$ the set $X^{ss}(L)$ is empty.

8.2 A G-linearized line bundle is called *G-effective* if $X^{ss}(L) \neq \emptyset$. Show that $L \otimes L'$ is G-effective if both L and L' are G-effective.

8.3 Let $\mathbb{G}_m$ act on an affine algebraic variety X and let $\mathcal{O}(X) = \bigoplus_{i\in\mathbb{Z}} \mathcal{O}(X)_i$ be the corresponding grading. Define $A_0 = \mathcal{O}(X)_0, A_{\geq 0} = \bigoplus_{i\geq 0} \mathcal{O}(X)_i$ and similarly $A_{\leq 0}, A_{>0}, A_{<0}$. Let $L \in \mathrm{Pic}^G(X)$ be trivial as a line bundle. Show that there are only three possibilities (up to isomorphism): $X^{ss}(L) = X, X \setminus V(I_+), X \setminus V(I_-)$, where I_+ (resp. I_-) is the ideal in $\mathcal{O}(X)$ generated by A_+ (resp. A_-). Show that in the first case $X^{ss}(L)/\!\!/\mathbb{G}_m$ is isomorphic to $\mathrm{Spm}(A_0)$, in the second (resp. the third) case $X^{ss}(L)/\!\!/\mathbb{G}_m$ is isomorphic to $\mathrm{Projm}(A_{\geq 0})$ (resp. $\mathrm{Projm}(A_{\leq 0})$).

8.4 In Example 8.6 show that the fibred product $\widetilde{Y} = Y_+ \times_{Y_0} Y_-$ is a nonsingular variety. Its projection to Y_0 is an isomorphism outside the origin, and the inverse image E of the origin is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Show that the restrictions of the projections from $\widetilde{Y}$ to $Y_\pm$ to E coincide with the two projection maps $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

8.5 Let G be a finite group acting regularly on X. Show that for any $L \in \mathrm{Pic}^G(X)$, $X^{ss}(L) = X^s(L)$. Also $X^s(L) = X$ if L is ample. Show that the assumption of ampleness is essential (even for the trivial group!).

8.6 Let $G = \mathrm{SL}_n$ act by conjugation on the affine space M_n of $n \times n$ matrices. Consider the corresponding action of G on the projective space $X = \mathbb{P}(M_n)$. Find the sets $X^{ss}(L), X^s(L)$ where $L \in \mathrm{Pic}^G(X)$.

8.7 Let $i : Y \hookrightarrow X$ be a closed G-invariant embedding, and let $L_Y = i^*(L)$ where L is an ample G-linearized line bundle on X. Assume that X is projective and G is linearly reductive, e.g. $\mathrm{char}(k) = 0$. Prove that, for any $y \in Y$,

$$\begin{aligned} y \in Y^s(i^*(L)) &\Leftrightarrow i(y) \in X^s(L), \\ y \in Y^s(i^*(L))_{(0)} &\Leftrightarrow i(y) \in X^s(L)_{(0)}. \end{aligned}$$

8.8 Consider Example 8.1 with $n = 3$ and $q_0 = 0, q_1 = 2, q_2 = 2, q_3 = 3$. Find all possible categorical quotients.