Chapter 7
The Permutahedron

7.1 Basic Definitions

The *permutahedron* is a very interesting polyhedron because its vertices and facets are in a one-to-one correspondence with, respectively, permutations and subsets of $[n]$ and therefore it shows the deep links that exist between abstract combinatorial structures and geometrical objects. The external representation of the permutahedron $P \subset \mathbb{R}^n$ is given by the following $2^n - 2$ inequalities plus one equality.

$$\begin{aligned} \sum_{i \in J} x_i &\geq \frac{|J|\,(|J|+1)}{2} \qquad J \subset [n],\; J \neq \emptyset,\, J \neq [n] \\ \sum_{i \in [n]} x_i &= \frac{n\,(n+1)}{2} \end{aligned} \tag{7.1}$$

Therefore every inequality is associated to a particular proper subset of $[n]$ and for the whole set $[n]$ we have an equality. Because of the equality the permutahedron is not full-dimensional. We first show that every inequality is facet-defining (clearly in the relative topology as explained at p. 10). Let S be a generic subset of $[n]$ and consider the point of coordinates

$$\hat{x}_i = \begin{cases} \dfrac{|S|+1}{2} & i \in S \\ \dfrac{n+|S|+1}{2} & i \notin S \end{cases}$$

Clearly $\sum_{i \in S} \hat{x}_i = |S|\,(|S|+1)/2$ and it is not difficult to verify that $\sum_{i \in [n]} \hat{x}_i = n\,(n+1)/2$. Let $J \neq S$. We may write

$$\frac{|S|+1}{2} = \frac{|J|+1+(|S|-|J|)}{2}, \qquad \frac{n+|S|+1}{2} = \frac{|J|+1+(n+|S|-|J|)}{2}$$

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so that

$$\sum_{i \in J} \hat{x}_i = \frac{|J|\,(|J|+1)}{2} + \frac{|J|\,(|S|-|J|) + n\,|J \setminus S|}{2}$$

If $|S| > |J|$ the second term in the sum is positive and then we have

$$\sum_{i \in J} \hat{x}_i > \frac{|J|\,(|J|+1)}{2} \tag{7.2}$$

If $|S| \leq |J|$, since $J \neq S$, we have $|J \setminus S| \geq 1$. So we have

$$\frac{|J|\,(|S|-|J|) + n\,|J \setminus S|}{2} \geq \frac{|J \setminus S|\,(n + |S| - |J|)}{2} > \frac{n+1-n}{2} = \frac{1}{2}$$

where the strict inequality comes from $|S| \geq 1$, $|J| < n$, $|J \setminus S| \geq 1$ and (7.2) holds also in this case.

Hence, equality holds only if $J = S$ and by Theorem 2.5 there is a one-to-one correspondence between a proper subset S of $[n]$ and a facet.

Let us now analyze the vertex structure. Each vertex is given by the intersection of at least n planes, of which one is the equality and the other $n-1$ are the inequalities satisfied as equalities. Take two of these inequalities and suppose they are associated to the subsets S and T. By summing them we get

$$\frac{|S|\,(|S|+1)}{2} + \frac{|T|\,(|T|+1)}{2} = \sum_{i \in S} x_i + \sum_{i \in T} x_i = \sum_{i \in S \cap T} x_i + \sum_{i \in S \cup T} x_i. \tag{7.3}$$

If we consider the subsets $S \cap T$ and $S \cup T$ we must have, by feasibility in (7.1) of the last expression in (7.3),

$$\frac{|S|\,(|S|+1)}{2} + \frac{|T|\,(|T|+1)}{2} \geq \frac{|S \cap T|\,(|S \cap T|+1)}{2} + \frac{|S \cup T|\,(|S \cup T|+1)}{2}.$$

It is not difficult to see that the following relation holds for any two sets S and T:

$$\frac{|S|\,(|S|+1)}{2} + \frac{|T|\,(|T|+1)}{2} \leq \frac{|S \cap T|\,(|S \cap T|+1)}{2} + \frac{|S \cup T|\,(|S \cup T|+1)}{2}$$

where the inequality is always strict except when $S \subset T$ or $T \subset S$. This means that the inequalities in (7.1) are satisfied as equalities only when they refer to subsets ordered by inclusion starting from a singleton set, adding one element at a time up to the whole set $[n]$, that corresponds to the equation $\sum_{i \in [n]} \hat{x}_i = n\,(n+1)/2$. This ordering can be viewed to as a permutation of $[n]$.

If we move away from a vertex along an edge, this corresponds to having one of the inequalities associated to the vertex no longer active. The active inequalities are still

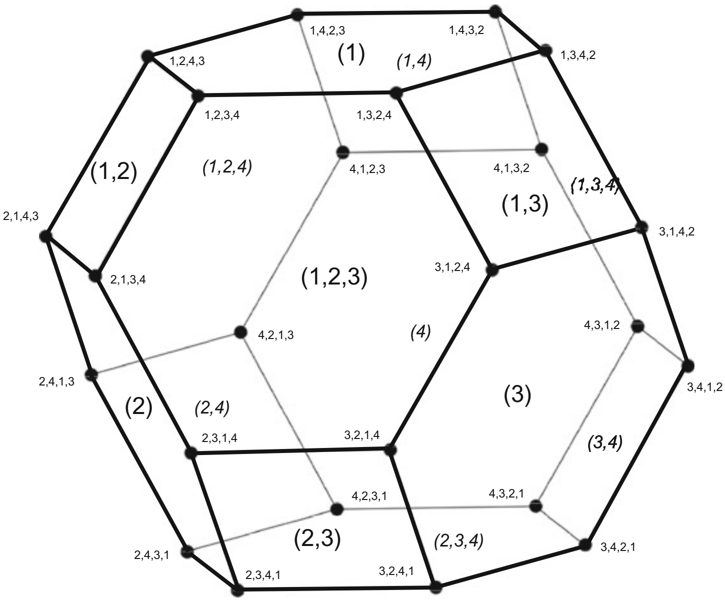

Fig. 7.1 The permutahedron for $n = 4$

ordered by inclusion but there is a jump of cardinality two in the ordering. If the vertex had in the ordering the three sets $\{a, b, c\}$, $\{a, b, c, d\}$ and $\{a, b, c, d, e\}$ (among others) and the second set is dropped along the edge, we have the jump from $\{a, b, c\}$ to $\{a, b, c, d, e\}$. When we reach the other vertex the only possibility to fill the jump and still have the subsets ordered by inclusion is to insert the subset $\{a, b, c, e\}$. Hence an adjacent vertex corresponds to a permutation obtained by switching two adjacent elements in the permutation.

In Fig. 7.1 we show the polyhedron for the case $n = 4$. It is a three dimensional object and hence it can be drawn and imagined as a real body. The visible edges are drawn thicker. Each vertex is labeled with a particular permutation of the set $\{1, 2, 3, 4\}$. Each facet is labeled with a particular proper subset of $\{1, 2, 3, 4\}$. The visible facets have a larger label in normal style, whereas the hidden facets have a smaller label in italics. For instance consider the vertex $\{3, 2, 1, 4\}$. It is generated by the intersection of the three planes corresponding to the sets $\{3\}$, $\{3, 2\}$ and $\{3, 2, 1\}$, that give the ordering $(3, 2, 1)$. The element 4 is the one missing and is added to the $n - 1$ elements to complete the permutation. Note the eight facets that are hexagons. They are permutahedra of smaller dimension, namely with $n = 3$.

In Fig. 7.8 at the end of the chapter we show a picture of a gadget in form of a permutahedron found in the shop of the Sinagoga del Agua in Ubeda, Andalusia, Spain.

7.2 A Compact Extended Formulation by LP Techniques

The permutahedron requires $2^n - 1$ constraints, i.e., an exponential number with respect to n. Since all inequalities are facet-defining it is not possible to represent the same polyhedron with a smaller number of inequalities. However, it turns out that it is possible to provide a compact extended formulation. Actually, we are going to provide three different formulations. We first describe a formulation via LP techniques as explained in Sect. 6.1.

The essential tool to build such a formulation is to have a separation problem that can be solved as an LP problem. Suppose we have a point $\bar{x}$ that satisfies $\sum_{i\in[n]} \bar{x}_i = n\,(n+1)/2$, and we want to know if there are violated inequalities in (7.1). The question is easily solved by sorting the $\bar{x}_i$ values in ascending order and checking, for each $k \in [n-1]$, if (where we assume the entries have been sorted)

$$\sum_{i\le k} \bar{x}_i \ge \frac{k\,(k+1)}{2} \tag{7.4}$$

This check can be modeled as the following $(n-1)$ LP problems, for $k \in [n-1]$,

$$\begin{aligned} \min \sum_{i\in[n]} & \bar{x}_i\, u_i \\ \sum_{i\in[n]} & u_i = k \\ -u_i & \ge -1 \qquad && i \in [n] \\ u_i & \ge 0 && i \in [n] \end{aligned}$$

because the solution is binary and $u_i = 1$ for the k indices corresponding to the first k smallest values of $\bar{x}_i$. The dual problems are

$$\begin{aligned} \max\ & k\, w_{0k} - \sum_{i\in[n]} w_{ik} \\ & w_{0k} - w_{ik} \le \bar{x}_i \qquad && i \in [n] \\ & w_{ik} \ge 0, && i \in [n] \end{aligned}$$

and the validity condition (7.4) becomes

$$k\, w_{0k} - \sum_{i\in[n]} w_{ik} \ge \frac{k\,(k+1)}{2}.$$

Therefore the permutahedron is the projection onto $\mathbb{R}^n$ of the polyhedron

$$\tilde{P} = \left\{ (x, w) \in \mathbb{R}^{n^2+n-1} : \begin{array}{ll} k\, w_{0k} - \sum_{i \in [n]} w_{ik} \geq \dfrac{k\,(k+1)}{2} & k \in [n-1] \\ w_{0k} - w_{ik} \leq x_i & i \in [n], k \in [n-1] \\ \sum_{i \in [n]} x_i = \dfrac{n\,(n+1)}{2} & \\ w_{ik} \geq 0 & i \in [n] \end{array} \right\}$$

There are $n + (n+1)(n-1) = n^2 + n - 1 = O(n^2)$ variables and $(n-1) + n(n-1) + 1 + n = n(n+1) = O(n^2)$ constraints. Hence $\tilde{P}$ is a compact extended formulation of the permutahedron. The dimension of $\tilde{P}$ is $n^2 + n - 2$, i.e., one less than that of the space where it is defined. To show this fact consider the point

$$x_i = \frac{n+1}{2}, i \in [n], \; w_{0k} = \frac{n+1}{2}, k \in [n-1], \; w_{ik} = \frac{k\,(n-k)}{2\,(n+1)}, i \in [n], k \in [n-1].$$

Then clearly $w_{0k} - w_{ik} < x_i$ and

$$k\, w_{0k} - \sum_{i \in [n]} w_{ik} = \frac{k\,(n+1)}{2} - n\,\frac{k\,(n-k)}{2\,(n+1)} = \frac{k}{2}\left(n + 1 - \frac{n\,(n-k)}{(n+1)}\right) =$$

$$\frac{k}{2}\left(\frac{n-k}{n+1} + (k+1)\right) > \frac{k\,(k+1)}{2}.$$

To get a slight idea of what is going on, we try to illustrate the situation for the simplest case, namely $n = 2$, the only one that can allow some drawing of the polyhedra. For $n = 2$ the permutahedron is just a segment in $\mathbb{R}^2$ connecting the points $(1, 2)$ and $(2, 1)$. Its compact extended formulation $\tilde{P}$ is a polyhedron of dimension four defined in $\mathbb{R}^5$. $\tilde{P}$ has five vertices whose coordinates $(x_1, x_2, w_{01}, w_{11}, w_{21})$ are respectively

$$(1, 2, 1, 0, 0), \quad (1, 2, 2, 1, 0), \quad (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 0, 0), \quad (2, 1, 1, 0, 0), \quad (2, 1, 2, 0, 1)$$

Note that there is a fractional vertex whose canonical projection onto the space of x variables falls within the segment. In Fig. 7.2 we show a bidimensional rendering of $\tilde{P}$. The underlying vertex-edges graph is complete, but this does not correspond to the general case. For $n = 3$ $\tilde{P}$ has 107 vertices and they do not have the same degrees: 88 vertices have degree 10, 18 have degree 19 and one vertex has degree 28.

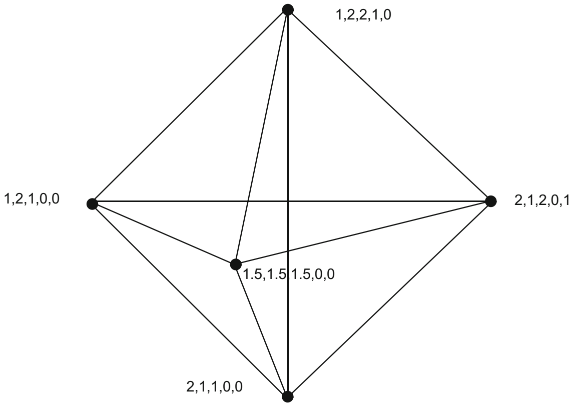

Fig. 7.2 The LP compact extended formulation of the permutahedron for $n = 2$

7.3 A Direct Compact Extended Formulation

We note that there is another simple compact extended formulation for the permutahedron, which can be simply derived by the properties of a permutation matrix. A permutation matrix is a 0–1 square matrix with exactly a 1 on each row and a 1 on each column. A permutation matrix can be seen as a point in $\mathbb{R}^{n^2}$. Clearly there are $n!$ permutation matrices of size n. The convex hull in $\mathbb{R}^{n^2}$ of the corresponding points is a polyhedron $\bar{Q}$ which is described by

$$\sum_{i\in[n]} x_{ij} = 1,\ j \in [n], \quad \sum_{j\in[n]} x_{ij} = 1,\ i \in [n], \quad x_{ij} \geq 0,\ i, j \in [n] \tag{7.5}$$

The dimension of $\bar{Q}$ is $(n-1)^2$ (there are n^2 variables subject to $2n - 1$ linearly independent equalities).

Now we build the following linear map $\mathbb{R}^{n^2} \to \mathbb{R}^n$, $x \mapsto \xi$

$$\xi_i = \sum_{j\in[n]} j\, x_{ij}, \qquad i \in [n] \tag{7.6}$$

Clearly each permutation matrix is transformed into a permutation of $[n]$. The Eq. (7.6) can also be viewed to as an injective linear map $x \mapsto (x, \xi)$ from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2+n}$. An injective linear map transforms a polyhedron into another polyhedron with the same dimension. Let $\bar{Q}$ be transformed into the polyhedron Q by the linear map, or, alternatively, $Q \subset \mathbb{R}^{n^2+n}$ is defined by (7.5) and (7.6). The permutahedron is the projection of Q onto $\mathbb{R}^n$.

With respect to the previous formulation this one is only slightly more expensive: one more variable and $n\,(n+3)$ constraints instead of $n\,(n+1)$. However, the extended polyhedron is dimensionally smaller and it does not exhibit fractional vertices.

7.4 A Minimal Compact Extended Formulation

A smaller compact extended formulation for the permutahedron can be derived by using a so-called *comparison network*, in particular a *sorting network*. A comparison network is made up of *wires* and *comparators*. A comparator has two input wires and two output wires (see Fig. 7.3) and works as follows: when two numbers x_1 and x_2 are fed as input, the comparator outputs two numbers that are $x_3 = \min\{x_1, x_2\}$ (above) and $x_4 = \max\{x_1, x_2\}$ (below).

A comparison network is obtained by assembling together a certain number of comparators in such a way that the output wires of a comparator can become input wires of other comparators (usually different otherwise one comparator would be useless). Some wires are only input wires and we consider them input wires of the network and some other wires are only output wires and we consider them as output wires of the network. It is easy to see that the number of input wires is equal to the number of output wires no matter how we assemble the comparators. If there are n input wires and m comparators then the total number of wires is $2\,m + n$. By construction the numbers in output are a permutation of the numbers in input and, similarly, the input numbers are a permutation of the output numbers.

We may ask which input permutations of $[n]$ are transformed by the network into the output $1, 2, \ldots, n$, i.e., the identity permutation. This set of permutations is called the feasible set for the network N. Let us denote this set as $F(N)$. If there are no comparators, $F(N)$ consists only of the identity. If there are enough comparators and they are placed in a right way, then we may have that $F(N)$ contains all permutations. In this case we speak of a *sorting network*.

See in Fig. 7.4 a sorting network with $n = 5$ and $m = 10$. The placement of the comparators mimics what a bubble sorting algorithm would do. The permutation $(2, 4, 3, 5, 1)$ is fed as input and in output there is the identity permutation. This placement requires $m = n\,(n-1)/2 = O(n^2)$ comparators for a sorting network with n input wires. It is not the most efficient way to build a sorting network. It has been proved that $O(n\,\log n)$ comparators are enough, although the hidden constant

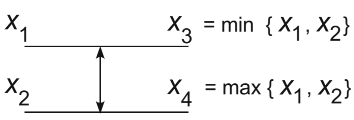

Fig. 7.3 A comparator

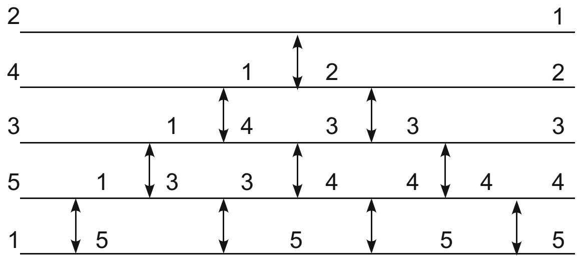

Fig. 7.4 A sorting network

in the big O notation is rather large. The reader can find in Ajtai et al. (1983) how to build an $O(n \log n)$ network.

Now we model the behavior of a comparison network as an LP problem. We associate to each wire a variable x_i, $i \in [2m + n]$. Let us number the variables so that $x_1, \ldots, x_n$ refer to the input wires and $x_{2m+1}, \ldots, x_{2m+n}$ (in this order from top to bottom) refer to the output wires. Refer to Fig. 7.5. For the comparator k, $k \in [m]$, let us denote the two input variables as $x_1(k)$ and $x_2(k)$ and the two output variables as $x_3(k)$ and $x_4(k)$. Then for each comparator, we impose the following constraints

$$x_3(k) \le x_1(k),\ \ x_3(k) \le x_2(k),\ \ x_1(k)+x_2(k) = x_3(k)+x_4(k),\ \ k \in [2m+n] \quad (7.7)$$

Note that (7.7) implies also $x_4(k) \ge x_1(k)$ and $x_4(k) \ge x_2(k)$. By assembling together all sets of constraints of type (7.7) and fixing the variables in output as $x_{2m+i} = i$, we define a polyhedron $Q \subset \mathbb{R}^{2m+n}$.

Let us now consider the polyhedron $\text{conv}(F(N)) \subset \mathbb{R}^n$, i.e., the convex hull of the permutations in $F(N)$. If $F(N)$ contains all permutations then $\text{conv}(F(N))$ is the permutahedron. It is possible to prove (see Conforti et al. 2010; Goemans 2015) that $\text{conv}(F(N))$ is the projection $\mathscr{P}(Q)$ of Q on the first n variables. Since Q is defined in $\mathbb{R}^{2m+n}$ and has $3\,(2m + n) + n$ constraints it is a compact extended formulation of the permutahedron. The number of facets of Q is $O(n \ \log n)$. Hence, by Theorem 2.10 this is a minimum formulation.

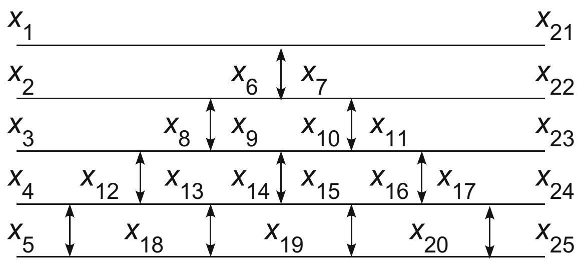

Fig. 7.5 The LP variables for a sorting network

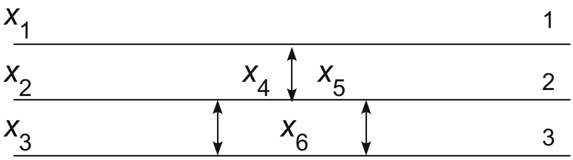

Fig. 7.6 A sorting network for $n = 3$

Note also that the dimension of Q is much less than $2\,m + n$. Since n variables are fixed and there are m linearly independent equations, the dimension of Q is m. Hence we can visualize Q for the case $n = 3$ that requires $m = 3$ comparators, like in Fig. 7.6, where the output variables have been already assigned their fixed values.

Since we have fixed the output variables, Q is actually embedded in $\mathbb{R}^6$ and not in $\mathbb{R}^9$. In Fig. 7.7 we see Q. The drawing has been obtained by projecting Q on a randomly chosen plane. The polyhedron Q has 8 vertices. Besides the six vertices, that are permutations of $(1, 2, 3)$, there are two more vertices, namely $(2, 2, 2, 2, 3, 2)$ and $(1, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2})$. We may be surprised by the existence of a vertex with fractional values for x_2 and x_3. A word of caution is necessary. It is not said that the coordinates $x_1, \ldots, x_n$ of the vertices of Q are integral. It is said that the coordinates of the projection of Q are integral. We have to recall that the permutahedron for $n = 3$ is bidimensional. Hence the fractional vertices go inside the projection and

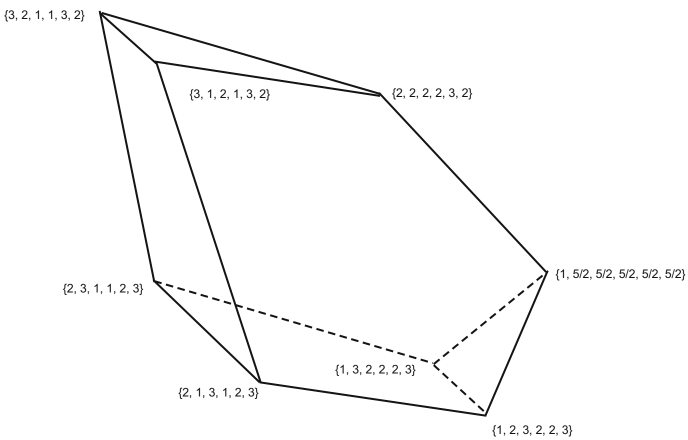

Fig. 7.7 The compact extended permutahedron for $n = 3$

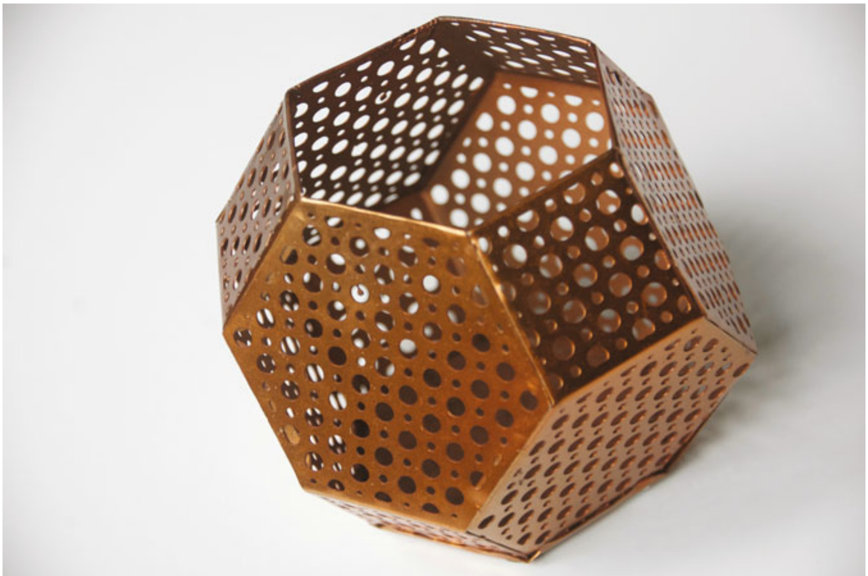

Fig. 7.8 A gadget in the form of a permutahedron

disappear.The result of the projection is the hexagon obtained by six edges of Q that join the six vertices that correspond to permutations. A seventh edge of this type disappears in the projection.