Birational geometry Winter 2020/2021

Problem set VIII: Torus actions, stability

In this series we contemplate the notion of stability and how it changes upon passing to open subsets.

1. This is well known so consider it as warm-up. Consider the action of $\mathbb{C}^*$ on $V = \mathbb{C}^3$ with weights $(1, 1, 1)$, that is

$$(\mathbb{C}^* \times \mathbb{C}^3) \ni (t, (x_1, x_2, x_3)) \to (tx_1, tx_2, tx_3) \in \mathbb{C}^3$$

(a) Find the set of stable points for this action. Find the grading induced by this action on the coordinate ring of V, $\mathbb{C}[V] = \mathbb{C}[x_1, x_2, x_3]$, and find the ring of invariants.

(b) Restrict the action to affine varieties $V_i = V \setminus \{x_i = 0\}$, for $i = 1, 2, 3$ and do the same as in the previous point. Note that the coordinate ring now is $\mathbb{C}[x_1, x_2, x_3, x_i^{-1}]$. Prove that now the rings of invariants determine geometric quotients $V_i \to V_i//\mathbb{C}^*$.

(c) Prove that on the quotients $V_i//\mathbb{C}^*$ over $V_i \cap V_j//\mathbb{C}^*$ glue to the projective space $\mathbb{P}^2$. Check that the induced map $V \setminus \{(0, 0, 0)\} \to \mathbb{P}^2$ is the geometric quotient.

2. Consider action of $\mathbb{C}^*$ on $V = \mathbb{C}^3$ with weights $(1, 1, -a)$ for some positive integer a, that is

$$(\mathbb{C}^* \times \mathbb{C}^3) \ni (t, (x_1, x_2, x_3)) \to (tx_1, tx_2, t^{-a}x_3) \in \mathbb{C}^3$$

(a) Find points of V which have nontrivial isotropy groups. Find the set of stable points for this action. Find the grading induced by this action on the coordinate ring of V. Find the ring of invariants, find its generators and their relations. Is the variety associated to the ring of invariants the geometric quotient?

(b) Restrict the action of $\mathbb{C}^*$ to $V_3 = V \setminus \{x_3 = 0\}$ and check if $V_3 \to V_3//\mathbb{C}^*$ is a geometric quotient. What is the quotient? You know this variety from previous exercises.

(c) Restrict the action of $\mathbb{C}^*$ to $V_i = V \setminus \{x_i = 0\}$ for $i = 1, 2$ and prove that $V_i \to V_i//\mathbb{C}^*$ is a geometric quotient. Prove that for $i = 1, 2$ there exists a morphism $V_i//\mathbb{C}^* \to V_3//\mathbb{C}^*$; is this morphism proper?

(d) Prove that there is a natural identification of quotients $V_i//\mathbb{C}^*$ for $i = 1, 2$ over $V_1 \cap V_2//\mathbb{C}^*$ and this identification gives rise to a variety which you know from previous sets of problems. Prove that the resulting morphism to $V_3//\mathbb{C}^*$ is proper; find the only nontrivial fiber of this morphism.

(e) What is the set of stable points of the action restricted to $V \setminus \{x_1 = x_2 = 0\}$? Prove that the variety obtained by glueing the two affine quotients $V_i//\mathbb{C}^*$, for $i = 1, 2$, from the previous point is the geometric quotient.

3. Consider the action of $\mathbb{C}^*$ on $V = \mathbb{C}^3$ with weights $(1, 1, -1, -1)$, that is

$$(\mathbb{C}^* \times \mathbb{C}^4) \ni (t, (x_1, x_2, x_3, x_4)) \to (tx_1, tx_2, t^{-1}x_3, t^{-1}x_4) \in \mathbb{C}^4$$

Do the analysis of this situation as in the previous problem, follow the same scheme of arguments as in points (a)–(e).