Birational geometry Winter 2020/2021

Problem set VI: Atiyah flop

The Atiyah flop is built on a small resolution of 3-dimensional quadric cone singularity

$$Z = \{(w_1, w_2, w_3, w_4) : w_1 w_4 = w_2 w_3\} \subset \mathbb{C}^4$$

By $\mathcal{O}(i, j)$ we denote the bundle $p_1^*\mathcal{O}(i) \otimes p_2^*\mathcal{O}(j)$ on the product $\mathbb{P}^1 \times \mathbb{P}^1$. Solving problem 2 is optional.

1. The quadric cone is an affine cone over 2-dimensional quadric in $\mathbb{P}^3$ which is the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$ with $w_1 = x_0 y_0,\ w_2 = x_0 y_1,\ w_3 = x_1 y_0, w_4 = x_1 y_1$ where x's and y's are homogeneous coordinates in $\mathbb{P}^1$'s.

(a) Prove that the complement of the vertex in Z has covering by $U_i = \{w_i \neq 0\}$ and each of them is isomorphic to $\mathbb{C}^2 \times \mathbb{C}^*$; find its coordinates.

(b) Prove that the divisor $D_0^x \subset Z$ which is the cone over line $\{x_0 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is not Cartier and it is defined by $w_1 = w_2 = 0$. Note that it is linearly equivalent to $D_1^x = \{w_3 = w_4 = 0\}$ and similar statements hold to $D_0^y = \{w_1 = w_3 = 0\}$ and $D_1^y = \{w_2 = w_4 = 0\}$. Prove that $D^x + D^y$ is Cartier.

(c) Prove that the blow-up of Z along divisor D_0^x is the closure of the graph of a rational map $Z \to \mathbb{P}^1$ defined by functions $[w_1, w_2]$. Note that in the product $\mathbb{C}^4 \times \mathbb{P}^1$ the resulting variety Z^x is an irreducible component dominating Z of the set defined by equations $w_1 w_4 = w_2 w_3$ and $w_1 z_2 = w_2 z_1$, where $[z_1, z_2]$ are coordinates in $\mathbb{P}^1$. Note that, in fact, Z^x is contained also in $w_3 z_2 = w_4 z_1$ which makes Z^x also the blow up of Z along the divisor D_1^x.

(d) Conclude that $Z^x \to \mathbb{P}^1$ is the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$, and therefore Z^x is a resolution of the quadric cone singularity with exceptional set $\mathbb{P}^1$. Prove that the strict tranform of divisor D^y has intersection 1 with the exceptional curve of the resolution while D^x has intersection -1.

2. Take $\mathbb{C}^3$ with coordinates (x, y, z) and take lines $L_x = V(x, z)$ and $L_y = V(x, y)$. Take the ideal $\mathcal{I} = (x, z) \cdot (y, z) = (xy, xz, yz, z^2)$ and consider $\widehat{Z}$ the blow up along $\mathcal{I}$ which is the closure of of the graph of the rational map

$$\mathbb{C}^3 \ni (x, y, z) \dashrightarrow [xy, xz, yz, z^2] = [u_1, u_2, u_3, u_4] \in \mathbb{P}^3$$

(a) Let E_x and E_y be irreducible divisors in $\widehat{Z}$ which dominate L_x and L_y, respectively. Prove that their intersection is a union of two lines in $\mathbb{P}^3$ meeting at a point where Z has quadric cone singularity discussed above.

(b) Prove that blowing first L_x and then the strict transform of L_y we get a resolution of the quadric cone singularity while the other the other order of blow-ups yields another small resolution, as above.

3. Let us consider polynomial ring $\mathbb{C}[x_0, x_1, y_0, y_1]$ with $\mathbb{Z}$ grading such that $\deg x_i = 1, \deg y_i = -1$. We have the decomposition in the graded pieces

$$A = \mathbb{C}[x_0, x_1, y_0, y_1] = \bigoplus_{d \in \mathbb{Z}} A^d$$

We consider ideals $\mathcal{I}^+ = (x_0, x_1)$ and $\mathcal{I}^- = (y_0, y_1)$ and subrings $A^+ = \bigoplus_{d \geq 0} A^d$ and $A^- = \bigoplus_{d \leq 0} A^d$.

(a) Prove that A^0 is a $\mathbb{C}$-algebra generated by $w_1 = x_0 y_0,\ w_2 = x_0 y_1,\ w_3 = x_1 y_0,\ w_4 = x_1 y_1$ and it is the coordinate ring of the quadric cone singularity hence $\operatorname{Spec} A^0 = Z$, where Z is as in the previous problems.

(b) Note that the mebedding $A^0 \hookrightarrow A$ determines a morphism of affine varieties in which the inverse image of the vertex is $V(\mathcal{I}^+ \cap \mathcal{I}^-)$ while over a general point the morphism has fibers $\mathbb{C}^*$ (in fact it is $\mathbb{C}^*$ bundle).

(c) Note that A^+ is generated over A^0 by x_0 and x_1 and therefore $Z^x = \operatorname{Proj} A^+$ has covering by two sets $W_i = \operatorname{Spec}(A^+_{x_i})^0$ for $i =, 0, 1$. Prove that $(A^+_{x_i})^0 \simeq \mathbb{C}[x_{1-i}/x_i, x_i y_0, x_i y_1]$ and the inclusion $\mathbb{C}[x_0, x_1] \hookrightarrow A^+$ yields a surjective morphism $\operatorname{Proj} A^+ \to \mathbb{P}^1$ which makes it the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$.

(d) Prove that the inclusion $A^0 \hookrightarrow A^+$ defines a small resolution $Z^x \to Z$ of the quadric cone singularity which was discussed in the previous exercises.

4. The construction which we discussed above and the resulting birational transformation

$$\begin{tikzcd} Z^x \arrow[rr, leftrightarrow, dashed] \arrow[dr] & & Z^y \arrow[dl] \\ & Z & \end{tikzcd}$$

is called Atiyah flop. Check that it has the following properties.

(a) All varieties are isomorphic in codimension 1, $Z^x,\ Z^y$ are smooth, Z is not $\mathbb{Q}$-factorial.

(b) The nontrivial fiber of $Z^* \to Z$ (denote it by C^x, C^y, respectively) is $\mathbb{P}^1$ with normal $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and by adjunction the canonical divisor is trivial on it.

(c) The strict transforms of divisors via the flop changes the sign of their intersection with C^x and C^y.