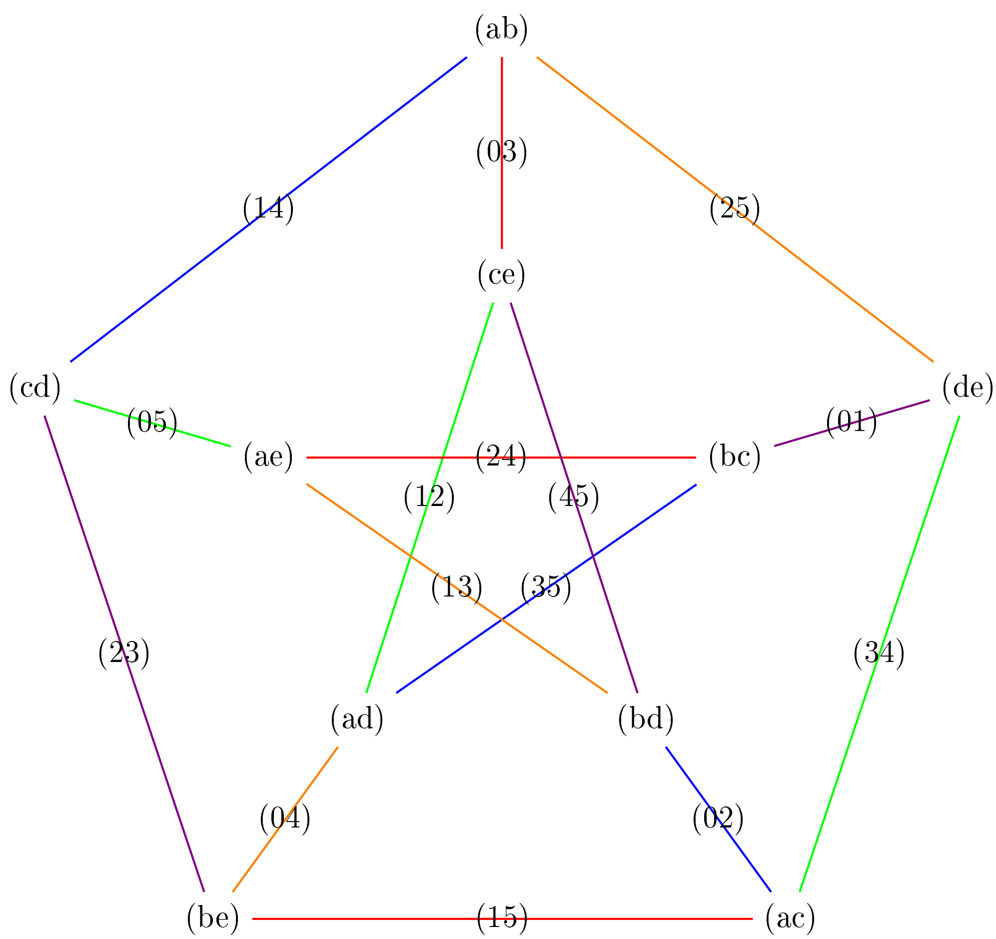
Birational geometry Winter 2020/2021

Problem set V: special del Pezzo surfaces and combinatorics
Reading: Yuri Manin, *Cubic forms* Ch. IV, sect. 24 and 25.

The following is the Petersen graph with 10 vertices labeled by pairs of letters in the set $\{a, \ldots, e\}$ and 15 edges labeled by pairs of digits in $\{0, \ldots, 5\}$; the edges are also colored by five colors.



1. Let us consider quadratic Cremona transformation of $\mathbb{P}^2$:

$$\mathbb{P}^2 \ni [z_1, z_2, z_3] \longrightarrow [z_1^{-1}, z_2^{-1}, z_3^{-1}] = [z_2 z_3, z_1 z_3, z_1 z_3] \in \mathbb{P}^2$$

(a) Prove that the Cremona is rational automorphism and an involution.

(b) Prove that blowing up $\mathbb{P}^2$ at $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ resolves the Cremona and it is the closure of the graph of Cremona in $\mathbb{P}^2 \times \mathbb{P}^2$. The blow up will be called $\mathbb{P}^2_3$; note that any three non-collinear points in $\mathbb{P}^2$ can be mapped by an automorphism of $\mathbb{P}^2$ to the three points above.

(c) Prove that Cremona determines biregular involutive automorphism of $\mathbb{P}^2_3$ and describe its action on the set of (-1) curves on $\mathbb{P}^2_3$. Find its fixed points in $\mathbb{P}^2_3$.

2. Let us consider blow up of $\mathbb{P}^2$ in four points, $\mathbb{P}^2_4$, no three of them collinear.

(a) Prove that (-1) curves on $\mathbb{P}^2_4$ are either the exceptional curves of the blowing up or strict transforms of lines passing through the four points which we blow up, so that $\mathbb{P}^2_4$ contains ten (-1) curves (you may use an exercise from the previous set).

(b) Prove that the incidence of (-1) curves on $\mathbb{P}^2_4$ is described by the Petersen graph in the following way that nodes stand for (-1) curves and edges indicate their transversal intersection.

(c) Prove that $\mathbb{P}^2_4$ admits five contractions to $\mathbb{P}^1$ which make $\mathbb{P}^2_4$ a conic fibration with three reducible fibers. The reducible conics are sums of (-1) curves which are indicated by the same color in the above Petersen graph. Note that each of five colors determines a partition of the set of digits 0–5 which is indicated by the labeling of edges.

(d) Prove that $\mathbb{P}^2_4$ admits five contractions to $\mathbb{P}^2$ each of them contracting four (-1) curves whose labels contain the same letter.

3. Automorphisms of $\mathbb{P}^2_4$.

(a) Prove that on the level of incidence diagrams of (-1) curves a contraction $\phi_{(\alpha\beta)} : \mathbb{P}^2_4 \to \mathbb{P}^2_3$ is obtained by removing the node $(\alpha\beta)$ associated to the (-1) curve which is contracted and also to removing three adjacent nodes so that the resulting diagram is a hexagon. Note that the edges joining the node $(\alpha\beta)$ with its neighbours are labeled by pairs in $\{0, \ldots, 5\}$ which make a partition of this set.

(b) Prove that the quadric Cremona tranformation as a biregular automorphism of $\mathbb{P}^2_3$ lifts up via $\phi_{(\alpha\beta)}$ to an involutive automorphism $\sigma_{(\alpha\beta)}$ of the del Pezzo surface $\mathbb{P}^2_4$. Note that $\sigma_{(\alpha\beta)}$ determines an involutive automorphism of the Petersen graph which we denote by the same name.

(c) Let Σ_4 be the group of permutation of the set $\{0 \ldots 4\}$ with $(\alpha\beta) \in \Sigma_4$ denoting the transposition $\alpha \leftrightarrow \beta$. Prove that the association $(\alpha\beta) \to \sigma_{(\alpha\beta)}$ determines a faithful representation of Σ_4 on the Petersen graph. Note that the action on the set of vertices can be identified as the action of Σ_4 by conjugation of the set of transpositions.

(d) We associate to a vertex of the above Petersen graph labeled by a transposition an element of Σ_5 (the group of permutation of of the set $\{0, \ldots, 5\}$) determined by partition of $\{0, \ldots, 5\}$ associated to the three edges stemming from that vertex. Prove that this defines an injective homomorphism of groups $\Sigma_4 \to \Sigma_5$ which is not conjugate to the standard embedding.

(e) Prove that the group of biregular automorphisms of $\mathbb{P}^2_4$ is generated by Cremona's $\sigma_{(\alpha\beta)}$ and it is isomorphic to Σ_4.