Birational geometry Winter 2020/2021

Problem set IIA
addendum to exercises II.1 and II.2.

All varieties are defined over an algebraically closed field k with $char(k) = 0$. In this series of problems X will be a complete normal variety (one may assume it even smooth, if this makes it easier) and L a line bundle over X trivialized over a covering $\mathcal{U} = (U_i)$ with transition functions $g_{ij} \in \mathcal{O}^*(U_{ij})$, with $U_{ij} = U_i \cap U_j$. By t_i we denote local coordinates on L associated to the covering $\mathcal{U}$, so that $t_i = g_{ij}t_j$ over U_{ij}.

1. We define $dlog(g_{ij}) = (dg_{ij})/g_{ij}$.

(a) Check that $(dlog(g_{ij}))$ defines a cocycle in $\mathrm{H}^1(\mathcal{U}, \Omega_X)$ and an element $c_1(L) \in \check{\mathrm{H}}^1(X, \Omega_X)$ in Čech cohomology, which we call the first Chern class of L.

(b) Prove that $c_1 : PicX = \mathrm{H}^1(X, \mathcal{O}^*) \to \mathrm{H}^1(X, \Omega_X)$ is a well defined homomorphism of groups.

(c) Verify that for a morphism of varieties $\varphi : Y \to X$ we have $\varphi^*(c_1(L)) = c_1(\varphi^*(L))$, where φ^* on the left-hand-side refers to the map of cohomology on Y associated to differential of the morphism φ, that is $\varphi^*\Omega_X \to \Omega_Y$.

(d) Prove that if $c_1(L) = 0$ then $L \equiv 0$. Hint: First note that it is enough to prove that ample divisors on curves have non-trivial c_1 and use the next exercise.

2. Let $f : C \to B$ be a degree d (finite) morphism of smooth complete curves.

(a) Prove that the morphism of sheaves of $\mathcal{O}_B$ modules $\mathcal{O}_B \to f_*\mathcal{O}_C$ splits.

(b) Prove that the induced morphism $\mathrm{H}^1(B, \Omega_B) \to \mathrm{H}^1(C, \Omega_C)$ is an ismorphism (remember that $char(k) = 0$).

3. Extensions of vector bundles. Suppose that we have an exact sequence of locally free sheaves of $\mathcal{O}_X$-modules over a complete variety X:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Prove that this sequence splits if and only if the associated boundary map $\mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^1(X, \mathcal{E})$ is zero.

Hint: For some covering U_i trivializing both $\mathcal{E}$ and $\mathcal{F}$ the transition matrices B_{ij} for $\mathcal{F}$ can be written as

$$B_{ij} = \begin{pmatrix} A_{ij} & s_{ij} \\ 0 & 1 \end{pmatrix}$$

where A_{ij} are transition matrices for $\mathcal{E}$ and (s_{ij}) can be identified with the Čech cocycle defining the image of the boundary map.

4. Let the situation be as above. The Euler-Atiyah extension of L is the extension

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{E}_L \longrightarrow \mathcal{O}_X \longrightarrow 0$$

defined by the class $c_1(L)$

(a) Prove that the Euler sequence over $\mathbb{P}^n$ is the Euler-Atiyah extension associated to $\mathcal{O}(d)$, where $d \neq 0$.

(b) Let (s_i) with $s_i \in \mathcal{O}_X(U_i)$ determine a section of L. Prove that (ds_i, s_i) determine a section of $\mathcal{E}_L \otimes L$ and we have a homomorphism $d : \mathrm{H}^0(X, L) \to \mathrm{H}^0(X, \mathcal{E}_L \otimes L)$ which splits the H^0 sequence for the Euler-Atiyah extension.

The bundle $\mathcal{E}_L \otimes L$ is called the bundle of first jets of L.

5. Let $L = \mathcal{O}_X(D)$, where D is an effective divisor. Recall that then $\mathrm{H}^0(X.L) = \{f \in K(X) : div(f) + D \geq 0\} \subset K(X)$, where $K(X)$ is the field of rational functions on X. This allows to define the structure of a graded k-algebra on the k-module $A_L = \bigoplus_{m \geq 0} \mathrm{H}^0(X, \mathcal{O}(mD))$ (this algebra does not have to be finitely generated over k).

(a) Prove that $M_L = \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^0(X, \mathcal{E} \otimes \mathcal{O}(mD))$ is a graded A_L-module.

(b) Prove that the induced map $d : A_L \to M_L$ is a differentiation.

6. Let $p : L^\bullet \to X$ be the associated k^*-bundle obtained by removing the zero section in the total space of L.

(a) Prove that we have an exact sequence of coherent sheaves over $L^\bullet$:

$$0 \longrightarrow p^*\Omega_X \longrightarrow \Omega_{L^\bullet} \longrightarrow \Omega_{L^\bullet/X} \longrightarrow 0$$

where the seaf on the right hand side is trivialized globally by a non-vanishing section dt_i/t_i.

(b) Prove that the push-forward of the above sequence via p_* yields an exact sequence of quasi-coherent sheaves over X:

$$0 \longrightarrow \bigoplus_m \Omega_X \otimes L^{\otimes m} \longrightarrow \bigoplus_m \mathcal{E}_L \otimes L^{\otimes m} \longrightarrow \bigoplus_m L^{\otimes m} \longrightarrow 0$$

where the decomposition may be defined in terms of the eigenspaces of the k^* action.