Birational geometry Winter 2020/2021

Problem set II:
differential forms, canonical divisor

Recall the following basic notions which can be found in the Hartshorne's book.
Differentials. Let A be a finitely generated k algebra (assume it is a domain). The A module Ω_A is generated by symbols da, with $a \in A$ modulo the relations: (1) $d(a+b) = da + db$, (2) $d(ab) = adb + bda$, (3) $d_{|k} = 0$. On a variety X we define the coherent sheaf Ω_X by glueing its values on the affine pieces.
Coherent sheaves on the projective space. Let $S = \bigoplus_{d \geq 0} S^d = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n = \operatorname{Proj} S$ which is covered by affine sets $U_i \simeq \mathbb{A}^n_k$ with coordinates $k[x_o/x_i, \ldots, x_n/x_i]$. Recall that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ we associate a graded S-module of its sections $\Gamma^*(\mathcal{F}) = \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j))$. On the other hand, for a graded S-module $M = \bigoplus_{j \in \mathbb{Z}} M^j$ we define a quasi-coherent sheaf $\widetilde{M}$ on $\mathbb{P}^n$ such that on every affine set U_i we have $\widetilde{M}(U_i) = M^0_{x_i} = \{m/x_i^r : m \in M^r\}$.

1. Euler sequence [Hartshorne, II.8]. In the notation introduced above, consider a morphism of graded S modules

$$\bigoplus_{j=0}^{n} S(-1) \cdot v_j \longrightarrow S$$

where v_j's are generators of degree 1 and each v_j is mapped to x_j. Let M be the kernel of this map.

(a) Show that we get the exact sequence of $\mathcal{O}$ sheaves

$$0 \longrightarrow \widetilde{M} \longrightarrow \mathcal{O}(-1)^{\oplus n+1} \longrightarrow \mathcal{O} \longrightarrow 0$$

(b) On U_i with coordinate ring $k[x_0/x_i, \ldots, x_n/x_i]$ we define a map $\psi_i : \Omega_{U_i} \to \mathcal{O}(-1)^{\oplus n+1}_{|U_i}$ by setting $\psi_i(d(x_r/x_i)) = (x_i v_r - x_r v_i)/x_i^2$. Prove that the map is isomorphism onto $\widetilde{M}_{|U_i}$.

(c) Prove that the homomorphisms ψ_i glue to isomorphism of sheaves $\Omega_{\mathbb{P}^n} \to \widetilde{M}$ so that we have the Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}(-1)^{\oplus n+1} \longrightarrow \mathcal{O} \longrightarrow 0$$

2. Prove the relative version of the Euler sequence: Let E be a vector bundle over smooth variety X. Consider its projectivisation $p : \mathbb{P}(E) \to X$. By $\Omega_{\mathbb{P}(E)/X}$ we denote the sheaf of relative differentials which are in the kernel of the map $Dp : p^*\Omega_X \to \Omega_{\mathbb{P}(E)}$. Then we have the following relative Euler sequence on $\mathbb{P}(E)$:

$$0 \longrightarrow \Omega_{\mathbb{P}(E)/X} \longrightarrow p^*E \otimes \mathcal{O}_{\mathbb{P}(E)}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

3. The canonical divisor K_X on a smooth variety X is defined as the determinant (top exterior power) of the cotangent sheaf (sheaf of differentials) Ω_X. If X is normal but not smooth then we define its canonical divisor as the Weil divisor which is the extension of the canonical divisor of the smooth part of X.

(a) Verify that the following statement makes sense, e.g. look in the Hartshorne book: If $I \lhd A$ is an ideal (may assume prime for simplicity) with the quotient ring $B = A/I$ then we have the following exact sequence of B modules

$$I/I^2 \longrightarrow \Omega_A \otimes B \longrightarrow \Omega_B \longrightarrow 0$$

where the class of $f \in I$ in I/I^2 is mapped with the first arrow to $df \otimes 1$ and the second arrow maps $da \otimes 1$ to the differential of the image of a in the residue ring $B = A/I$.

(b) Suppose that $Y \subset X$ is a smooth prime effective divisor in a smooth variety X. Prove that the above sequence yields the following exact sequence of sheaves on Y:

$$0 \longrightarrow \mathcal{O}(-Y)_{|Y} \longrightarrow \Omega_{X|Y} \longrightarrow \Omega_Y \longrightarrow 0$$

(c) In the situation of the previous point show the adjunction formula $K_Y \simeq (K_X + Y)_{|Y}$.

(d) Prove that the canonical divisor on $\mathbb{P}^1$ has degree -2; hint: show that the form dx/x (where x a non-homogeneous coordinate) has two simple poles.

(e) Use adjunction to calculate the canonical divisor on $\mathbb{P}^n$, for $n \geq 2$; hint: use adjunction and the fact that the resctriction of divisors to a hyperplane $\operatorname{Pic} \mathbb{P}^n \longrightarrow \operatorname{Pic} \mathbb{P}^{n-1}$ is an isomorphism.

(f) Calculate the canonical divisor of a smooth hypersurface of degree d in $\mathbb{P}^n$.

(g) Use the relative Euler sequence to calculate the canonical divisor of $\mathbb{P}(E)$ in terms of K_X, $detE$, rkE and $\mathcal{O}_{\mathbb{P}(E)}(1)$.

4. Canonical divisor for rational morphisms.

(a) Let $p : \widetilde{X} \to X$ be a blow-up of a smooth variety X of dimension n at a smoothth subvariety $Z \subset X$ of codimension r. Prove that $K_{\widetilde{X}} \sim p^*(K_X) + (r-1)E_Z$, where E_Z is the exceptional divisor of the blow-up.

(b) Let $\varphi : X \to Y$ be a birational morphism of two projective smooth varieties. Prove that $K_X \sim \varphi^* K_Y + \sum_i a_i E_i$, where E_i are exceptional divisors of φ and $a_i > 0$ for all i. The number a_i are called discrepancy of φ on E_i.

(c) Suppose that X and Y are smooth surfaces. Let $C \subset X$ be a curve which is not contracted by φ. Conclude that $\varphi(C)^2 \geq C^2$ and $\varphi(C) \cdot K_Y \leq C \cdot K_X$.

5. Let $F : X \longrightarrow Y$ be a finite morphism of smooth varieties. Prove that $K_X \sim F^* K_Y + R_F$ where $R_F \subset X$ is an effective divisor called the ramification of F. Use this fact to prove that smooth every curve dominated by $\mathbb{P}^1$ is isomorphic to $\mathbb{P}^1$.