Birational geometry Winter 2020/2021

Problem set XII: Quotients of quadrics

In this set we analyse torus action on quadrics. The notation is consistent with that explained in the lecture. In particular $\mathbb{T}^r$ acts on a projective variety X with fixed point components Y_i. In addition we will use the definition of compass: Suppose that X is smooth and $y \in Y_i$ is a fixed point. Then we have a natural linearization of the action of $\mathbb{T}^r$ on the cotangent space $T_yX^* = \mathfrak{m}_y/\mathfrak{m}_y^2$. This determines the set of weights of this action $\nu_1(y), \dots, \nu_d(y) \in M$ (possibly with repetitions, $d = \dim X$) which is the same for every y in the component Y_i, hence denoted by $\nu_1(Y_i), \dots, \nu_d(Y_i)$. The number of zero weights among $\nu_i(Y)$ is equal to $\dim Y$. The set of non-zero $\nu_i(Y)$'s, with possibly multiple entries, will be called *the compass* of the component Y_i in X with respect to the action of $\mathbb{T}^r$. Reading: `https://arxiv.org/pdf/1802.05002.pdf`, pp. 10–20, Example 2.20.

Torus $\mathbb{T}^r = (\mathbb{C}^*)^r$ with coordinates $(t_1, \dots, t_r)$ acts on $\mathbb{C}^{2r+1}$ as follows:

$$(t_1, \dots, t_r) \cdot (z_0, z_1, z_2, \dots, z_{2r-1}, z_{2r}) = (z_0, t_1 z_1, t_1^{-1} z_2, \dots, t_r z_{2r-1}, t_r^1 z_{2r})$$

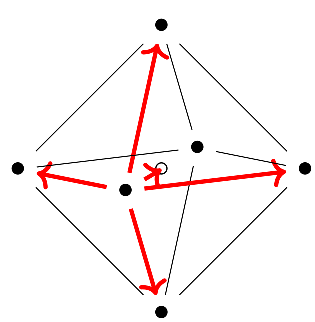
The action of $\mathbb{T}^r$ descends to an almost faithful action (i.e. with only finite kernel) on the quadric $\mathbb{Q}^{2r-1} \subset \mathbb{P}^{2r}$ given by the equation

$$z_0^2 + z_1 z_2 + \dots + z_{2r-1} z_{2r} = 0.$$

The action has $2r$ isolated fixed points. If M is the lattice of characters of $\mathbb{T}^r$ with the basis $e_1, \dots, e_r$, then

$$\Delta(\mathbb{Q}^{2r-1}, \mathcal{O}(1), \mathbb{T}^r) = \mathrm{conv}(\pm e_1, \dots, \pm e_r)$$

The compass of $\mathbb{T}^r$ at the fixed point associated to the character e_i consists of $-e_i$ and $\pm e_j - e_i$, for $j \neq i$. Note that the compass generates the semigroup $\mathbb{R}_{\geq 0}(\Delta - e_i) \cap M$. The picture here is for $r = 3$.



The case of even dimensional quadric is similar. The torus $\mathbb{T}^r$ acts on $\mathbb{C}^{2r}$

$$(t_1, \ldots, t_r) \cdot (z_1, z_2, \ldots, z_{2r-1}, z_{2r}) = (t_1 z_1, t_1^{-1} z_2, \ldots, t_r z_{2r-1}, t_r^{-1} z_{2r})$$

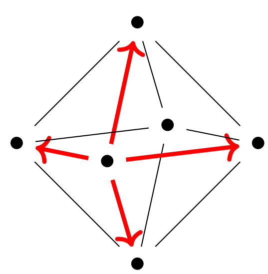
The action of $\mathbb{T}^r$ descends to an action of the quotient torus $\mathbb{T}^r / \langle(-1, \ldots, -1)\rangle$ on the quadric $\mathbb{Q}^{2r-2} \subset \mathbb{P}^{2r-1}$ given by equation

$$z_1 z_2 + \cdots + z_{2r-1} z_{2r} = 0$$

The action has $2r$ isolated fixed points. As before, $M = \mathbb{Z}^r$ generated by e_i's and $M' \subset M$ is an index 2 sublattice of vectors $\sum_i a_i e_i$ such that $\sum_i a_i$ is even. Now

$$\Delta(\mathbb{Q}^{2r-2}, \mathcal{O}(1), \mathbb{T}^r) = \mathrm{conv}(\pm e_1, \ldots, \pm e_r)$$

and the compass of $\mathbb{T}^r$ at the fixed point associated to the character e_i consists of $\pm e_j - e_i$, for $j \neq i$. Note that the compass generates $\mathbb{R}_{\geq 0}(\Delta - e_i) \cap M'$.



Do the following problems for both odd and even case. If you find hard solving higher dimensional cases then concentrate on the case $r = 3$ and use the octahedral pictures from above. Also use the symmetry of the picture.

1. Classify r-dimensional orbits of $\mathbb{T}^r$-action. Classify them in terms of polytopes Δ and in terms of their finite isotropy group. If possible, write the equation of an orbit of the given type. Find the degree of a general orbit with respect to the line bundle $\mathcal{O}(1)$. Discuss possible degenerations of a general orbit into $\mathbb{T}^r$ invariant subsets (union of orbits).

2. Classify geometric quotients. Find which linearizations yield geometric quotients, describe them in terms of the polytope $\Gamma = \Delta$. Remember to replace $\mathcal{O}(1)$ by its multiple or to pass to the affine situation (quadric cone) with an action of $\mathbb{T}^{r+1}$. Find the semistable locus and describe the quotient.