Ample vector bundles and introduction to Mori theory

Problem set V, for November 13th

All varieties are defined over an algebraically closed field $k$.
Recall the following basic notions which can be found in the Hartshorne’s book.

Differentials. Let $A$ be a finitely generated $k$ algebra (assume it is a domain). The $A$ module $\Omega_A$ is generated by symbols $da$, with $a \in A$ modulo the relations: (1) $d(a + b) = da + db$, (2) $d(ab) = adb + bda$, (3) $d|_k = 0$. On a variety $X$ we define the coherent sheaf $\Omega_X$ by glueing its values on the affine pieces.

Coherent sheaves on the projective space. Let $S = \bigoplus_{d \geq 0} S^d = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n = \text{Proj} S$ which is covered by affine sets $U_i \simeq \mathbb{A}^n_k$ with coordinates $k[x_0/x_i, \ldots, x_n/x_i]$. Recall that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ we associate a graded $S$-module of its sections $\Gamma^*(\mathcal{F}) = \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j))$. On the other hand, for a graded $S$-module $M = \bigoplus_{j \in \mathbb{Z}} M^j$ we define a quasi-coherent sheaf $\mathcal{M}$ on $\mathbb{P}^n$ such that on every affine set $U_i$ we have $\mathcal{M}(U_i) = M^0_{x_i} = \{m/x_i^r : m \in M^r\}$.

1. Euler sequence [Hartshorne, II.8]. In the notation introduced above, consider a morphism of graded $S$ modules

$$\bigoplus_{j=0}^n S(-1) \cdot v_j \longrightarrow S$$

where $v_j$’s are generators of degree 1 and each $v_j$ is mapped to $x_j$. Let $M$ be the kernel of this map.

(a) Show that we get the exact sequence of $\mathcal{O}$ sheaves

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}(-1)^{\oplus_{n+1}} \longrightarrow \mathcal{O} \longrightarrow 0$$

(b) On $U_i$ with coordinate ring $k[x_0/x_i, \ldots, x_n/x_i]$ we define a map $\psi_i : \Omega_{U_i} \rightarrow \mathcal{O}(-1)_{|U_i}^{\oplus_{n+1}}$ by setting $\psi_i(d(x_r/x_i)) = (x_i v_r - x_r v_i)/x_i^2$. Prove that the map is isomorphism onto $\mathcal{M}_{|U_i}$. 

(c) Prove that the homomorphisms $\psi_i$ glue to isomorphism of sheaves $\Omega_{\mathbb{P}^n} \rightarrow \widetilde{M}$ so that we have the Euler sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n} \rightarrow \mathcal{O}(-1)^{\oplus n+1} \rightarrow \mathcal{O} \rightarrow 0$$

2. Find the connecting (boundary) homomorphism $H^0(\mathbb{P}^n, \mathcal{O}) \rightarrow H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n})$ in the sequence of cohomology of the Euler sequence; calculate it using the covering $(U_i)$ and express it in terms of Čech cohomology.

3. For $d > 0$ consider the twisted Euler sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n} \otimes \mathcal{O}(d) \rightarrow \mathcal{O}(d-1)^{\oplus n+1} \rightarrow \mathcal{O}(d) \rightarrow 0$$

Prove that the $H^0$ of this sequence is exact and that it actually splits by $\mathcal{O}(d) \ni s \rightarrow \sum (\partial s/\partial x_i) \cdot v_i \in \mathcal{O}(d-1)^{\oplus (n+1)}$.

4. Use Euler sequence to calculate cohomology of sheaves $\Omega^p_{\mathbb{P}^n}$ of $p$-forms on $\mathbb{P}^n$. Hint: prove the following lemma. Given an exact sequence of vector bundles $0 \rightarrow V \rightarrow W \rightarrow L \rightarrow 0$, with $L$ of rank 1, for any positive $p$ we get the exact sequence $0 \rightarrow \Lambda^p V \rightarrow \Lambda^p W \rightarrow \Lambda^{p-1} V \otimes L \rightarrow 0$.

5. Prove the relative version of the Euler sequence: Let $E$ be a vector bundle over smooth variety $X$. Consider its projectivisation $p : \mathbb{P}(E) \rightarrow X$. By $\Omega_{\mathbb{P}(E)/X}$ we denote the sheaf of relative differentials which are in the kernel of the map $Dp : p^* \Omega_X \rightarrow \Omega_{\mathbb{P}(E)}$. Then we have the following relative Euler sequence on $\mathbb{P}(E)$:

$$0 \rightarrow \Omega_{\mathbb{P}(E)/X} \rightarrow p^* E \otimes \mathcal{O}_{\mathbb{P}(E)}(-1) \rightarrow \mathcal{O} \rightarrow 0$$

Use the relative Euler sequence to calculate the canonical divisor of $\mathbb{P}(E)$ in terms of $K_X$, $detE$, $rkE$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$. 