CHAPTER 2. Rational scrolls

This chapter describes scrolls, especially the rational normal scrolls. These varieties occur throughout projective and algebraic geometry, and the student will never regret the investment of time studying them. One reason for presenting them here is that they can be discussed with very little background, and can be used to illustrate many constructions of algebraic geometry with substantial examples. I use them here to give simple examples of rational surfaces, K3 surfaces, elliptic surfaces, and surfaces with pencils of curves of genus 2, 3, 4, etc.

In intrinsic terms, a scroll is a $\mathbb{P}^{n-1}$-bundle $F \to C$ over a curve C, that is, an algebraic fibre bundle, isomorphic to $U_i \times \mathbb{P}^{n-1}$ over small Zariski open sets $U_i \subset C$, and glued by transition functions given by morphisms $U_i \cap U_j \to \mathrm{PGL}(n)$. It can be written as the projectivisation $F = \mathbb{P}(E)$ of a vector bundle E, and the study of general scrolls is essentially equivalent to that of vector bundles over curves. In the case $C = \mathbb{P}^1$ everything is much simpler, because the base curve $\mathbb{P}^1$ is a very explicit object, and every vector bundle is a direct sum of line bundles. Thus any question about scrolls can be solved in very explicit terms. I give some examples in the text, and many more in the exercises (see for example Ex. 2.6–2.9).

As well as discussing scrolls, this section introduces and gives examples of the following notions: linear system, free linear system, very ample linear system and projective embedding, quadrics of rank 3 and 4, determinantal variety, base locus of linear system, divisor class group $\operatorname{Pic} X$, intersection numbers, Veronese surface and cones over it, vector bundles over curves, projectivised bundle $\mathbb{P}_C(E)$, Chern numbers, Harder–Narasimhan filtration, K3 surface, elliptic surface, Weierstrass normal form, surface with a pencil of curves of genus $g = 1, 2, 3, 4, \dots$.

Summary

(1) $\mathbb{F} = \mathbb{F}(a_1, \dots, a_n)$ is defined as a quotient of the $(n+2)$-dimensional space $(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^n \setminus 0)$ by an action of two copies of the multiplicative group $\mathbb{G}_m$. There is a projection morphism $\pi\colon \mathbb{F} \to \mathbb{P}^1$ making $\mathbb{F}$ into a $\mathbb{P}^{n-1}$ fibre bundle.

(2) Rational functions on $\mathbb{F}$ are defined as ratios of bihomogeneous polynomials. $\mathbb{F}$ has an embedding into $\mathbb{P}^N$ with the fibres $\mathbb{P}^{n-1}$ of π mapping to $(n-1)$-planes of $\mathbb{P}^N$.

(3) The divisor class group of $\mathbb{F}$ can be generated by two elements, $\operatorname{Pic} \mathbb{F} = \mathbb{Z}L \oplus \mathbb{Z}M$, where L is a fibre of π and M is a relative hyperplane.

(4) $\mathbb{F}$ contains negative subscrolls $B_c = \mathbb{F}(a_k, \dots, a_n)$ corresponding to the "unstable" filtration of the integers $a_1, \dots, a_n$, that is, when $a_1, \dots, a_{k-1} \ge c > a_k, \dots, a_n$.

(5) The base locus of the linear system $|aL + bM|$ is determined in terms of negative subscrolls $B_c \subset \mathbb{F}$; nonsingularity conditions on the general $D \in |aL+bM|$ impose combinatorial conditions on the numerical data, and often lead to finite lists.

(6) Applications of scrolls: varieties in $\mathbb{P}^n$ of small degree, del Pezzo's theorem, Castelnuovo varieties.

(7) Fibred surfaces in scrolls.

2.1. Reminder: $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$

I start by recalling the product of two projective spaces as treated in elementary textbooks (compare, for example, [Sh], Chapter I or [UAG], §5), which is a very useful analogy for scrolls. $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$ can be defined as the quotient of $(\mathbb{C}^l \setminus 0) \times (\mathbb{C}^m \setminus 0)$ by the action of two copies of the multiplicative group $\mathbb{C}^* \times \mathbb{C}^*$ acting separately on the two factors:

$$(x_1, \dots, x_l; y_1, \dots, y_m) \mapsto (\lambda x_1, \dots, \lambda x_l; \mu y_1, \dots, \mu y_m) \qquad \text{for } (\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*.$$

Subvarieties of $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$ are defined by *bihomogeneous* polynomials, that is, polynomials that are homogeneous separately in $x_1, \dots, x_l$ and $y_1, \dots, y_m$, and rational functions on $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$ as quotients of two bihomogeneous polynomials of the same bidegree. Next $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$ has the Segre embedding into usual projective space

$$\mathbb{P}^{l-1} \times \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{lm-1}$$

defined by bilinear forms

$$(x_1, \dots, x_l; y_1, \dots, y_m) \mapsto \big(u_{ij} = x_i y_j\big)_{\substack{i=1\dots l \\ j=1\dots m}}.$$

The image is defined by equations $\operatorname{rank}(u_{ij}) \le 1$.

Remark. As an algebraic geometer, I should say that $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1}$ is the quotient of the variety $(\mathbb{A}^l \setminus 0) \times (\mathbb{A}^m \setminus 0)$ by the action of the algebraic group $\mathbb{G}_m \times \mathbb{G}_m$. Here $\mathbb{A}^l$ is affine space, the variety corresponding to k^l for a field k, and $\mathbb{G}_m$ is the algebraic group corresponding to the multiplicative group k^*. If it makes life simpler, you can replace $\mathbb{A}^l$ by $\mathbb{C}^l$ and $\mathbb{G}_m$ by $\mathbb{C}^*$ throughout.

2.2. Definition of $\mathbb{F}(a_1, \dots, a_n)$

Let $a_1, \dots, a_n$ be integers. I define the scroll $\mathbb{F} = \mathbb{F}(a_1, \dots, a_n)$ as the quotient of $(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^n \setminus 0)$ by an action of $\mathbb{G}_m \times \mathbb{G}_m$, the product of two multiplicative groups. Write t_1, t_2 for coordinates on $\mathbb{A}^2$ and $x_1, \dots, x_n$ on $\mathbb{A}^n$, and λ and μ for elements of the two factors of $\mathbb{G}_m \times \mathbb{G}_m$, that is, $(\lambda, \mu) \in \mathbb{G}_m \times \mathbb{G}_m$. The action is given as follows:

$$\begin{aligned} (\lambda, 1)&\colon (t_1, t_2; x_1, \dots, x_n) \mapsto (\lambda t_1, \lambda t_2; \lambda^{-a_1} x_1, \dots, \lambda^{-a_n} x_n); \\ (1, \mu)&\colon (t_1, t_2; x_1, \dots, x_n) \mapsto (t_1, t_2; \mu x_1, \dots, \mu x_n). \end{aligned}$$

Note first that the ratio $t_1 : t_2$ is preserved by the action of $\mathbb{G}_m \times \mathbb{G}_m$, so that the projection to the first factor defines a morphism $\pi\colon \mathbb{F} \to \mathbb{P}^1$:

$$\begin{array}{ccc} (\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^n \setminus 0) & \longrightarrow & \mathbb{F}(a_1, \dots, a_n) \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ (\mathbb{A}^2 \setminus 0) & \longrightarrow & \mathbb{P}^1 \end{array}$$

Remark. Compared to 2.1, I have restricted to the case $l = 2$ (so that the first factor is $\mathbb{P}^1$), and generalised the group action to allow it to mix up the two factors, so that $\mathbb{F} \to \mathbb{P}^1$ can be a nontrivial $\mathbb{P}^{n-1}$ fibre bundle. The material of 2.2–7 follows exactly the remaining steps of 2.1.

2.3. $\mathbb{F}(a_1, \ldots, a_n)$ as a fibre bundle

Above any given ratio $(t_1 : t_2) \in \mathbb{P}^1$, I can normalise to fix the values of t_1, t_2 with the given ratio, and this takes care of the action of the first factor $\mathbb{G}_m$; after this, the fibre of π over $(t_1 : t_2)$ consists of the set of ratios $(x_1 : \cdots : x_n)$, forming a copy of $\mathbb{P}^{n-1}$. Thus every fibre of the projection map $\pi\colon \mathbb{F} \to \mathbb{P}^1$ is isomorphic to $\mathbb{P}^{n-1}$. As I show in Theorem 2.5 below, a good 19th century way of understanding $\mathbb{F}$ is to embed it in a projective space so that the fibres of π are linearly embedded $(n-1)$-planes.

Remark. $\pi\colon \mathbb{F} \to \mathbb{P}^1$ is an example of a *fibre bundle* with fibre $\mathbb{P}^{n-1}$ and structure group the diagonal subgroup of $\mathrm{PGL}(n)$. More explicitly, on the affine piece $U_0 = (t_2 \neq 0) \subset \mathbb{P}^1$, I set $t_2 = 1$, so that $\pi^{-1}(U_0) = \mathbb{A}^1 \times \mathbb{P}^{n-1}$, with $t_1 = t_1/t_2$ the affine coordinate in the first factor and $(x_1 : \cdots : x_n)$ homogeneous coordinates in the second. Similarly, over $U_\infty = (t_1 \neq 0)$ I get $\pi^{-1}(U_\infty) = U_\infty \times \mathbb{P}^{n-1}$. The affine coordinates $t_1 = t_1/t_2$ and $t_2 = t_2/t_1$ on U_0 and U_∞ are related on the overlap $U_0 \cap U_\infty$ in the usual way by $t_1 = 1/t_2$, and the two open sets $\pi^{-1}(U_0) = U_0 \times \mathbb{P}^{n-1}$ and $\pi^{-1}(U_\infty) = U_\infty \times \mathbb{P}^{n-1} \subset \mathbb{F}$ are glued together as follows:

$$t_1 : 1; x_1^{(0)} : \cdots : x_n^{(0)} \xrightarrow{t_1^{-1}} 1, 1/t_1; t_1^{a_1} x_1^{(0)} : \cdots : t_1^{a_n} x_n^{(0)} = 1, t_2; x_1^{(\infty)} : \cdots : x_n^{(\infty)}.$$

(The arrow is the action of $\lambda = t_1^{-1} \in \mathbb{G}_m$). In brief, $\mathbb{F}$ is the union of two copies of $\mathbb{A}^1 \times \mathbb{P}^{n-1}$ glued together by $t_1 \mapsto t_1^{-1}$ in the first factor and $\mathrm{diag}(t_1^{a_1}, \ldots, t_1^{a_n})$ in the second.

2.4. Bihomogenous polynomials

Rational functions on $\mathbb{F}$ are defined as ratios of bihomogenous polynomials, that is, eigenfunctions of the action of $\mathbb{G}_m \times \mathbb{G}_m$. I write down some vector spaces of bihomogeneous functions. I've already given one, the space $\langle t_1, t_2 \rangle$ of homogeneous polynomials of degree 1 in t_1, t_2 and degree 0 in $x_1, \ldots, x_n$; the ratio $t_1 : t_2$ defines the projection $\pi\colon \mathbb{F} \to \mathbb{P}^1$.

Next, consider the functions that are linear in $x_1, \ldots, x_n$. It's clear that this means that the second factor of $\mathbb{G}_m \times \mathbb{G}_m$ acts by $(1, \mu)\colon h \mapsto \mu h$. Consider polynomials that are also invariant under the action of the first factor, that is, $(\lambda, 1)\colon h \mapsto h$. Obviously, to cancel the group action $x_i \mapsto \lambda^{-a_i} x_i$, the linear term x_i must be accompanied by a monomial $t_1^b t_2^c$ with $b + c = a_i$, and hence the vector space of μ-invariant polynomials is based by

$$S^{a_1}(t_1, t_2) x_1, \ldots, S^{a_n}(t_1, t_2) x_n,$$

where $S^a(t_1, t_2) = \{t_1^a, t_1^{a-1} t_2, \ldots, t_2^a\}$ is the set of monomials of degree a in t_1, t_2. Of course, $S^a = \emptyset$ if $a < 0$ and $S^0 = \{1\}$. The notation S^a stands for the ath symmetric tensor power: if $\langle \Sigma \rangle$ denotes the vector space spanned by a set Σ and Sym^a the ath symmetric tensor power of a vector space then $\langle S^{a_1}(t_1, t_2) \rangle = \mathrm{Sym}^a \langle t_1, t_2 \rangle$.

In the same way, the space of polynomials that are linear in $x_1, \ldots, x_n$ and in the λ^e eigenspace of the first factor is based by

$$S^{a_1+e}(t_1, t_2) x_1, \ldots, S^{a_n+e}(t_1, t_2) x_n;$$

its dimension is $\sum^+_{i=1,\ldots,n} (a_i + e + 1)$, where $\sum^+$ means you only take the sum of the terms that are ≥ 0.

There is a similar description of the bihomogeneous polynomials of any bidegree, that is, of degree d in $x_1, \dots, x_n$ and *extra degree* e in the t_i; this is the vector space based by the monomials $t_1^{e_1} t_2^{e_2} x_1^{d_1} \cdots x_n^{d_n}$ with $\sum d_i = d$, and $e_1 + e_2 = \sum_{i=1}^n d_i a_i + e$. I discuss this in more detail later.

2.5. Theorem (Linear embeddings $\mathbb{F} \hookrightarrow \mathbb{P}^N$). *Suppose that $a_1, \dots, a_n > 0$; then the ratios between the bihomogeneous polynomials*

$$S^{a_1}(t_1, t_2) x_1, \dots, S^{a_n}(t_1, t_2) x_n \tag{2.5.1}$$

define an embedding $\varphi \colon \mathbb{F}(a_1, \dots, a_n) \hookrightarrow \mathbb{P}^N$ (where $N = \sum_{i=1}^n (a_i + 1) - 1$) in such a way that every fibre $\mathbb{P}^{n-1}$ of π goes into a linearly embedded $(n-1)$-plane.

The image is the subvariety of $\mathbb{P}^N$ defined by the determinantal equations

$$\operatorname{rank} \begin{pmatrix} u_1 & u_2 & \dots & u_{a_1} & u_{a_1+2} & \dots & u_{a_1+a_2+1} & \dots & u_N \\ u_2 & u_3 & \dots & u_{a_1+1} & u_{a_1+3} & \dots & u_{a_1+a_2+2} & \dots & u_N + 1 \end{pmatrix} \le 1.$$

The matrix here has n blocks of size $2 \times a_i$; in each block, the $(1, j)$th entry for $j \ge 2$ repeats the $(2, j-1)$st entry. The meaning of the determinantal equation is that if the monomials (2.5.1) are listed as $t_1^{a_1} x_1, t_1^{a_1 - 1} t_2 x_1, \dots, t_1^{a_2} x_2, \dots$ then the ratio $t_1 : t_2$ equals the ratio between the first and second rows, that is

$$\frac{t_1}{t_2} = \frac{t_1^{a_1} x_1}{t_1^{a_1-1} t_2 x_1} = \frac{t_1^{a_1-1} t_2 x_1}{t_1^{a_1-2} t_2^2 x_1} = \dots = \frac{t_1^{a_2} x_2}{t_1^{a_2-1} t_2 x_2} = \dots = \frac{t_1 t_2^{a_n - 1} x_n}{t_2^{a_n} x_n}.$$

Proof. The proof is very similar to that for the Segre embedding $\mathbb{P}^{l-1} \times \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{lm-1}$ (see [Sh], Chapter I).

$\mathbb{F}$ is covered by a number of open sets $U_{ij} : (t_i \neq 0, x_j \neq 0)$ for $i = 1, 2$ and $j = 1, \dots n$, each isomorphic to $\mathbb{A}^n$. The piece U_{11} is typical. The n affine coordinates on it are t_2/t_1 and $t_1^{a_i - a_1} x_i / x_1$ for $i = 2, \dots, n$.

The set of monomials include $t_1^{a_1} x_1, t_1^{a_1-1} t_2 x_1, t_1^{a_2} x_2, \dots, t_n^{a_n} x_n$. The first of these is nonzero everywhere on U_{11}, so that the ratio is well defined there. The n affine coordinates of U_{11} are precisely given by the ratios between $t_1^{a_1} x_1$ and the n succeeding monomials, so that these embed U_{11}.

In the given determinantal equations, clearly if $u_1 = 1$ then all the remaining u_i are determined by u_2 and $u_{a_1+2}, u_{a_1+a_2+3}, \dots$ corresponding to the n affine coordinates of U_{11}. Q.E.D.

Remarks. (a) "Linear generation" of scrolls. The image variety $\mathbb{F}(a_1, \dots, a_n) \subset \mathbb{P}^N$ has the following description in projective geometry. Consider a fixed copy of $\mathbb{P}^1$ with homogeneous coordinates t_1, t_2, and n embeddings $v_{a_i} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^{a_i}$ defined by

$$(t_1 : t_2) \mapsto (t_1^{a_i} : t_1^{a_i - 1} t_2 : \dots : t_2^{a_i});$$

this is the a_ith *Veronese embedding*, and the image $\Gamma_i = v_{a_i}(\mathbb{P}^1)$ is called the *rational normal curve* of degree a_i. Embed all the projective spaces $\mathbb{P}^{a_i} \hookrightarrow \mathbb{P}^N$ as linearly independent subspaces of a common $\mathbb{P}^N$ with $N = \sum_{i=1}^n (a_i + 1) - 1$. Now the curves Γ_i are all identified with $\mathbb{P}^1$, so that it makes sense to take the linear span of corresponding points. This is $\mathbb{F}(a_1, \dots, a_n) \subset \mathbb{P}^N$; prove this using the determinantal equations as an exercise. (See Ex. 2.14.)

(b) If the assumption of the theorem is weakened to $a_i \geq 0$ then the ratio of the monomials (2.5.1) still defines a morphism $\varphi\colon \mathbb{F} \to \mathbb{P}^N$ that embeds each fibre of π as an $(n-1)$-plane, but if some $a_i = 0$ then x_i appears only in a single monomial $S^{a_i}(t_1, t_2)x_i = \{x_i\}$, and $\varphi(\mathbb{F})$ is a cone. The determinantal equations still make perfectly good sense, but the coordinate corresponding to x_i does not appear in the equations. (See Ex. 2.15.)

(c) "Linear equations of scrolls". There is a classical description of the determinantal equations of Theorem 2.5 as $c = \operatorname{codim}(\mathbb{F} \subset \mathbb{P}^N)$ quadrics through a $(N-2)$-plane. (See Ex. 2.16.)

2.6. Particular cases

$\mathbb{F}(1,0)$ is a surface scroll, $\varphi\colon \mathbb{F}(1,0) \to \mathbb{P}^2$ is the blowup of a point. More generally $\varphi\colon \mathbb{F}(1,0,\dots,0) \to \mathbb{P}^n$ is the pencil of hyperplanes through a given codimension 2 linear subspace.

$\mathbb{F}(2,1) \subset \mathbb{P}^4$ is the cubic scroll.

$\mathbb{F}(1,1) \cong \mathbb{P}^1 \times \mathbb{P}^1 \cong Q \subset \mathbb{P}^3$ is the nonsingular quadric surface with a choice of projection.

$\mathbb{F}(2,0) \to Q' \subset \mathbb{P}^3$ is the standard resolution of the ordinary quadric cone (blowup).

More generally $\mathbb{F}(1,1,\underbrace{0,\dots,0}_{n-2}) \to Q_4 \subset \mathbb{P}^{n+1}$ is a resolution of a quadric of rank 4 associated with a chosen family of generators, and $\mathbb{F}(2,\underbrace{0,\dots,0}_{n-1}) \to Q_3 \subset \mathbb{P}^{n+1}$ is the standard resolution of a quadric of rank 3.

$\varphi\colon \mathbb{F}(a,0) \to \overline{\mathbb{F}}_a \subset \mathbb{P}^a$ is the blowup of the cone over a rational normal curve. The surface scroll $\mathbb{F}(a,0) \cong \mathbb{F}(a+b,b)$ for any $b \in \mathbb{Z}$ is usually called $\mathbb{F}_a$. As I discuss below, the exceptional curve of the resolution $B = \varphi^{-1}(0) \subset \mathbb{F}_a$ is a section of $\pi\colon \mathbb{F}_a \to \mathbb{P}^1$ with $B^2 = -a$.

$\mathbb{F}(0,\dots,0) \cong \mathbb{P}^1 \times \mathbb{P}^{n-1}$ with $\varphi\colon \mathbb{F}(0,\dots,0) \to \mathbb{P}^{n-1}$ the second projection. This is the only case with all $a_i \geq 0$ for which φ is not birational.

2.7. Lemma. *The divisor class group of the scroll $\mathbb{F}$ is the free Abelian group*

$$\operatorname{Pic} \mathbb{F} = \mathbb{Z}L \oplus \mathbb{Z}M,$$

with generators the following two divisor classes: L is the class of a fibre of π, and M the class of any monomial $t_1^b t_2^c x_i$ with $b + c = a_i$. (If all the $a_i > 0$, then M is the divisor class of the hyperplane section under the embedding $\mathbb{F} \subset \mathbb{P}^N$ of Theorem 2.5.)

Proof. First note that any two fibres of $\pi\colon \mathbb{F} \to \mathbb{P}^1$ are linearly equivalent: because a fraction $\alpha(t_1,t_2)/\beta(t_1,t_2)$, where α, β are linear forms, is a rational function on $\mathbb{F}$ with divisor the difference of two fibres. Thus the divisor class L of a fibre is well defined.

To see M more clearly, let $F_i \subset \mathbb{F}$ be the locus defined by $x_i = 0$; this is clearly the subscroll $F_i = \mathbb{F}(a_1,\dots,\widehat{a_i},\dots,a_n)$. Then the divisors $a_iL + F_i$ are all linearly equivalent, and define the divisor class M. Indeed, the fraction $t_1^{a_i}x_i/t_1^{a_j}x_j$ is a rational function on $\mathbb{F}$ with divisor $(a_iL + F_i) - (a_jL + F_j)$.

L and M are linearly independent in $\operatorname{Pic}\mathbb{F}$, since if $aL + bM \overset{\text{lin}}{\sim} 0$ then restricting to any fibre $\mathbb{P}^{n-1}$ of π gives $b = 0$, and then clearly $a = 0$. Finally, I have to prove that every divisor of $\mathbb{F}$ is linearly equivalent to $aL + bM$ for some $a, b \in \mathbb{Z}$. Indeed, any irreducible codimension 1 subvariety $C \subset \mathbb{F}$ is defined by a single bihomogeneous polynomial equation in the sense of 2.4; to see this, take the inverse image in $\mathbb{A}^2 \times \mathbb{A}^n$, and argue as in the case of usual projective space. If C is defined by f with given bidegree, it is obvious how to fix up a monomial $t_1^{a_1 d+e} x_1^d$ with the same bidegree, so that $f/t_1^{a_1 d+e} x_1^d$ is a rational function, and $C \overset{\text{lin}}{\sim} eL+dM$. Q.E.D.

Remark. In this notation, the canonical class of $\mathbb{F}(a_1, \dots, a_n)$ is given by

$$K_{\mathbb{F}} \overset{\text{lin}}{\sim} -2L - \sum F_i \overset{\text{lin}}{\sim} (-2 + \sum a_i)L - nM.$$

See A.10 and Ex. A.13 for details.

2.8. Negative subscrolls $B_b \subset \mathbb{F}$ and the base locus of linear systems

Linear systems on general varieties are discussed below. Here I treat from an elementary point of view the linear system $|eL + dM|$ on $\mathbb{F}$, the family of divisors of $\mathbb{F}$ parametrised by the vector space of bihomogeneous polynomials of degree d in the x_i and extra degree e in the t_i. I assume $d \geq 1$.

Definition. The *subscroll* corresponding to a subset $\{a_{i_1}, \dots, a_{i_m}\} \subset \{a_1, \dots, a_n\}$ is the subvariety $\mathbb{F}(a_{i_1}, \dots, a_{i_m}) \subset \mathbb{F}(a_1, \dots, a_n)$ defined by the equations $x_j = 0$ for $j \notin \{a_{i_1}, \dots, a_{i_m}\}$. It is clearly a scroll in its own right with bihomogeneous coordinates $t_1 : t_2; x_{i_1} : \dots : x_{i_m}$.

For any b, define the *negative subscroll* $B_b \subset \mathbb{F}(a_1, \dots, a_n)$ to be the subscroll corresponding to the subset $\{a_i \mid a_i \leq b\}$. Suppose now for convenience that $a_1 \leq a_2 \leq \dots \leq a_n$. Then

$$B_b = \mathbb{F}(a_1, \dots, a_m) \subset \mathbb{F}(a_1, \dots, a_n),$$

where m is determined by $a_1 \leq \dots \leq a_m \leq b < a_{m+1} \leq \dots \leq a_n$.

As shown by (1) of the following theorem, the point of the definition is that the B_b have a tendency to be base locuses of linear systems.

Theorem. *(1) The base locus of $|-(b+1)L + M|$ is exactly B_b.*

(2) Suppose that $b = a_m$. Then B_b is contained with multiplicity $< \mu$ in the base locus of $|eL + dM|$ if and only if

$$e + bd + (a_n - b)(\mu - 1) \geq 0. \tag{2.8.1}$$

Proof. (1) An element of the linear system $|eL + M|$ is a hypersurface in $\mathbb{F}$ defined by a form f which is a sum of monomials $S^{a_i+e}(t_1, t_2)x_i$ for $i = 1, \dots, n$. Clearly, if $a_i \leq b < -e$ then x_i doesn't appear in any such monomial; therefore f vanishes identically on the locus $x_{m+1} = \dots = x_n = 0$. I told you so!

(2) The proof of (2) will make more sense after thinking about the worked examples 2.10–11 and drawing the corresponding Newton polygons.

An element of $|eL + dM|$ is defined by a bihomogeneous polynomial of bidegree d, e. Monomials having degree $\geq \mu$ in $x_{m+1}, \dots, x_n$ vanish with multiplicity μ along B_b. Thus the assertion is that there exists a monomial of bidegree d, e of degree $< \mu$ in $x_{m+1}, \dots, x_n$ if and only if (2.8.1) holds. To make a monomial of extra degree e, the term $x_m^{d-\mu+1} x_n^{\mu-1}$ must be accompanied by a monomial of degree

$$e + a_m(d - \mu + 1) + a_n(\mu - 1),$$

which equals the left-hand side of (2.8.1), since $b = a_m$. This is obviously the highest accompanying degree of any of the allowed monomials. Q.E.D.

2.9. Special case: the surface scroll $\mathbb{F}_a = \mathbb{F}(0, a)$

Here $a \geq 0$, and $\mathbb{F}_a = \mathbb{F}(0, a)$ is the surface scroll. I adopt the notation $B = B_0 : (x_2 = 0)$ for the negative section (the point $(1, 0)$ on every fibre $\mathbb{P}^1$), and $A = L$ for the fibre. Then $\operatorname{Pic} \mathbb{F}_a = \mathbb{Z}A \oplus \mathbb{Z}B$, and the intersection numbers are

$$A^2 = 0, AB = 1, B^2 = -a.$$

Proof. From Theorem 2.5, the morphism $\varphi \colon \mathbb{F}_a \to \overline{\mathbb{F}}_a \subset \mathbb{P}^a$ defined by $|M| = |aA + B|$ is the natural resolution of the cone over the rational normal curve of degree a, with B contracting to the vertex. The curve $(x_1 = 0) : M \overset{\text{lin}}{\sim} aA + B$ maps to a hyperplane section, and is disjoint from B. Hence $B(aA+B) = 0$. Since $A^2 = 0$ and $AB = 1$ are obvious, this completes the proof. Q.E.D.

In this case $|eL+dM| = |(e+ad)A+dB|$, so that the conclusions of Theorem 2.8, (1) and (2) have very simple interpretations:

$$B \text{ is fixed in } |(e+ad)A + dB| \iff e < 0 \iff B\big((e+ad)A + dB\big) < 0$$

and

$$\begin{aligned} \mu B \text{ is fixed in } |(e+ad)A + dB| &\iff e + a(\mu - 1) < 0 \\ &\iff B\big((e+ad)A + (d - \mu + 1)B\big) < 0. \end{aligned}$$

2.10. Worked example: the Maroni invariant of a trigonal curve

A curve C (of genus $g \geq 3$, assumed to be nonhyperelliptic) is *trigonal* if it has a 3-to-1 map $C \to \mathbb{P}^1$, or equivalently, if it has a g^1_3, a free linear system $|D|$ with $\dim |D| = 1$ and $\deg D = 3$. Consider the canonical model $C \subset \mathbb{P}^{g-1}$. Then geometric RR says that 3 points $P_1 + P_2 + P_3$ on C move in a g^1_3 if and only if they are collinear in $\mathbb{P}^{g-1}$. (Compare 3.2 and [4 authors], Chapter III, §3.) It follows at once that the canonical model of a trigonal curve is contained in a rational normal surface scroll $C \subset \mathbb{F}(a_1, a_2) \subset \mathbb{P}^{g-1}$ where $g = a_1 + a_2 + 2$ (or cone $\overline{\mathbb{F}(0, a_2)}$), and the pencil $|A|$ on $\mathbb{F}$ cuts out the g^1_3 on C.

Order the a_i as $a_1 \leq a_2$, set $a = a_2 - a_1$, and, as before, write A for the fibre of $\mathbb{F}_a \to \mathbb{P}^1$ and $B \subset \mathbb{F}_a$ for the negative section. Then $\mathbb{F}(a_1, a_2) \subset \mathbb{P}^{g-1}$ is $\mathbb{F}_a$ embedded by $a_2A + B$. The canonical curve $C \subset \mathbb{F}_a$ is linearly equivalent to $\alpha A + 3B$ for some α, and computing the degree gives $\alpha = a + a_2 + 2$. By Theorem 2.8, the surface scroll $\mathbb{F}_a$ contains a nonsingular curve $C \subset \mathbb{F}_a$ linearly equivalent to $\alpha A + 3B$ if and only if $\alpha - 3a = B(\alpha A + 3B) \geq 0$. Therefore $\alpha \geq 3a$, that is, $a + a_2 + 2 \geq 3a$, which works out finally as

$$3a \leq g + 2, \quad \text{or} \quad 3a_2 \leq 2g - 2, \quad \text{or} \quad 3a_1 \geq g - 4.$$

Thus the quantity a is a further invariant of trigonal curves, the *Maroni invariant*. The final inequality says in particular that $\mathbb{F}(a_1, a_2)$ can only be a cone if $g = 4$.

2.11. Worked example: elliptic surfaces $X \subset \mathbb{F} = \mathbb{F}(a_1, a_2, a_3)$ and Weierstrass fibration

I give a typical application of Theorem 2.8. Let $\mathbb{F} = \mathbb{F}(a_1, a_2, a_3) \to \mathbb{P}^1$ be a 3-fold scroll with $a_1 \le a_2 \le a_3$ and $X \subset \mathbb{F}$ a surface meeting the general fibre of $\mathbb{F} \to \mathbb{P}^1$ in a nonsingular cubic curve. Then $X \in |(k+2-\sum a_i)L + 3M|$ for some $k \in \mathbb{Z}$; note that the class of X is unchanged if I change $(a_1, a_2, a_3) \mapsto (a_1 - \nu, a_2 - \nu, a_3 - \nu)$ for some $\nu \in \mathbb{Z}$ and $M \mapsto M - \nu L$. To tidy up the calculation, I will assume later that $a_1 = 0$. (The class of X is arranged so that the canonical class of X is $K_X = kL_{|X}$, by the adjunction formula, compare A.11 below.)

I write out the equation of X as a relative cubic

$$\sum_{i+j+k=3} c_{ijk}(t_1, t_2) x_1^i x_2^j x_3^k,$$

and keep track of the degrees $\deg c_{ijk} = (k+2-\sum a_i) + ia_1 + ja_2 + ka_3$ of the accompanying homogeneous terms in the Newton polygon:

$$\begin{array}{ccccccc} & & & k+2+2a_1-a_2-a_3 & & & \\ & & k+2+a_1-a_3 & & k+2+a_1-a_2 & & \\ & k+2+a_2-a_3 & & k+2 & & k+2-a_2+a_3 & \\ k+2-a_1+2a_2-a_3 & & k+2-a_1+a_2 & & k+2-a_1+a_3 & & k+2-a_1-a_2+2a_3 \end{array} \qquad (*)$$

In order for X to be nonsingular at the general fibre of $\mathbb{F} \to \mathbb{P}^1$, its base locus is restricted by two conditions: $B_{a_2} \not\subset X$ and $2B_{a_1} \not\subset X$. These conditions just say that the general fibre of $X \to \mathbb{P}^1$ does not break up as the line $x_3 = 0$ plus a conic (so at least one of the degrees on the left-hand side of the Newton polygon is ≥ 0, that is, $k+2-a_1+2a_2-a_3 \ge 0$), and does not have $(1,0,0)$ as a double point (so that at least one of the degrees in the top corner is ≥ 0, that is, $k+2+a_1-a_2 \ge 0$).

The criterion of Theorem 2.8 is of the form

$$(k+2-\sum a_i) + 3b + (a_n - a_m)(\mu - 1) \ge 0,$$

which works out as follows:

Condition $B_{a_2} \not\subset X$:. $b = a_2$, $\mu = 1$, so that $(k+2-\sum a_i) + 3a_2 + 0(a_3 - a_2) \ge 0$; setting $a_1 = 0$ and repeating the usual assumptions on the a_i gives

$$a_2 + k + 2 \ge a_3 - a_2 \ge 0. \qquad (1)$$

Condition $2B_{a_1} \not\subset X$:. $b = a_1$, $\mu = 2$, so that $k+2-\sum a_i + 3a_1 + (a_3 - a_1) \ge 0$; setting $a_1 = 0$ gives

$$k+2 \ge a_2 \ge 0. \qquad (2)$$

For fixed value of k, (1) and (2) have solutions

$$a_2 = 0, \dots, k+2;$$
$$a_3 = a_2, \dots, 2a_2 + k + 2.$$

It is fun to consider the extreme cases of these inequalities. Referring to the Newton polygon, one sees that:

(1) If $k+2 < a_2 + a_3$ then the curve $B_{a_1} \subset \operatorname{Bs}|(k+2-\sum a_i)L + 3M|$.

(2) If $k+2 < a_3$ then every $X \in |(k+2-\sum a_i)L + 3M|$ contains B_{a_1} and is tangent along it to B_{a_2}. In this case, the general X has singularities on B_{a_1} at the $k+2-a_2$ zeros of $c_{201}(t_1,t_2)$.

(3) If $k+2 < a_3 - a_2$ then every $X \in |(k+2-\sum a_i)L + 3M|$ contains B_{a_1} and has a flex along B_{a_2}.

The extreme case of the inequalities (1–2) are $a_2 = k+2$, $a_3 = 3(k+2)$. In this case the critical coefficients of x_2^3 and $x_1^2x_3$ are homogeneous forms in t_1, t_2 of degree zero, that is constants, so that X has equation

$$1 \cdot x_2^3 + 1 \cdot x_1^2 x_3 + \text{other terms}.$$

In other words, $X \subset \mathbb{F}$ is a nonsingular surface, with every fibre the Weierstrass normal form of an elliptic curve.

2.12. Final remarks on scrolls

1. Minimal degree. Scrolls occur throughout projective algebraic geometry as projective varieties of minimal degree: del Pezzo's theorem (from the early 1880s) says that an irreducible d-dimensional variety V spanning $\mathbb{P}^n$ has degree $\geq n-d+1$, and if equality holds then V is a linearly embedded scroll (as in Theorem 2.5), a cone over a scroll (as in Remark 2.5, (b)), or one of the sporadic cases: $\mathbb{P}^n$ itself, a quadric hypersurface $Q \subset \mathbb{P}^n$, the Veronese surface $W \subset \mathbb{P}^5$ or a cone over W. See [Bertini] or Eisenbud and Harris [E–H] for proofs of different vintages, or do it for yourself (Ex. 2.19).

Hypersurfaces in scrolls play a similar role in the study of curves whose degree is small compared to the genus, or surfaces of general type with K^2 small compared to p_g. Compare Ex. 2.24 or [Harris] or [4 authors], Chapter 3 (including the exercises).

2. Surfaces with a pencil of curves. Many surfaces come with a natural pencil of curves of small genus; for example, Castelnuovo and Horikawa showed that surfaces with $K^2 = 3p_g - 7$ for which the canonical map φ_K is birational (and $p_g \geq 7$) are naturally relative quartics in a scroll $\varphi_K(X) \subset \mathbb{F} \subset \mathbb{P}^{p_g-1}$. These surfaces can therefore be studied as hypersurfaces in an explicit rational 3-fold. Compare Ex. 2.24–25.

3. Scrolls over curves of genus ≥ 1. An n-dimensional scroll can more generally be defined as a $\mathbb{P}^{n-1}$-bundle $F \to C$ over any curve C, that is, a fibre bundle with fibre $\mathbb{P}^{n-1}$ and structure group $\mathrm{PGL}(n)$. It can be proved that every scroll F is the projectivisation $F = \mathbb{P}(\mathcal{E})$ of a rank n vector bundle $\mathcal{E}$ over C (compare Tsen's theorem in C.4 below). The assumption that the base curve is $\mathbb{P}^1$ is a major simplifying feature, which makes it possible to give a completely elementary self-contained treatment: in this case there is no effort in saying what the base curve $\mathbb{P}^1$ and the vector bundle $\mathcal{E}$ is: every vector bundle over $\mathbb{P}^1$ is a direct sum of $\mathcal{O}_{\mathbb{P}^1}(a_i)$. (This is a famous theorem, traditionally attributed to Grothendieck, Atiyah, Birkhoff, Hilbert, Dedekind, Gauss, Euler, Archimedes, …).

However, for the knowledgeable reader, essentially each part of the discussion here carries over to the more general case. This was one of the prime motivations of the theory of algebraic vector bundles over curves in the 1950s. The positive subsheaves $\bigoplus_{a_i \geq c} \mathcal{O}_{\mathbb{P}^1}(a_i)$ that correspond to the negative sections of the scrolls generalise to the Harder–Narasimhan filtration of $\mathcal{E}$, etc. See for example [H1], Chapter V, §2.

Exercises to Chapter 2

1. Prove that $\mathbb{F}(a) \cong \mathbb{P}^1$ for any $a \in \mathbb{Z}$.

2. Prove that $\mathbb{F}(0,0) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Generalise to $\mathbb{F}(0,\dots,0)$ (with n zeros).

3. Show how to cover $\mathbb{F}(a_1,\dots,a_n)$ by $2n$ standard affine pieces isomorphic to $\mathbb{A}^n$, and write down the transition functions glueing any two pieces.

4. Prove that $\mathbb{F}(3,1) \cong \mathbb{F}(2,0)$ by comparing coordinates patches, and that $\mathbb{F}(2,0) \cong \mathbb{F}(1,-1)$.

5. Prove that in general

$$\mathbb{F}(a_1,\dots,a_n) \cong \mathbb{F}(a_1+b,\dots,a_n+b)$$

for any $b \in \mathbb{Z}$. [Hint: Every element of the group $\mathbb{C}^* \times \mathbb{C}^*$ can be written as a product of $(\lambda,1)$ and $(1,\mu)$ (for suitable λ,μ), or alternatively as a product of (λ,λ^b) and $(1,\mu')$ (for suitable λ,μ'). In other words, the two actions of $\mathbb{C}^* \times \mathbb{C}^*$ only differ by an automorphism.] Deduce that the assumption $a_1 = 0$ is harmless if you're only interested in $\mathbb{F}$ up to isomorphism.

How is $\mathbb{F}(1,1) \cong \mathbb{F}(-1,-1)$ reconciled with Theorem 2.5?

6. Use the description of $\operatorname{Pic}\mathbb{F}$ and Theorem 2.8, (1) to prove that

$$\mathbb{F}(a_1,\dots,a_n) \cong \mathbb{F}(b_1,\dots,b_n) \iff \{a_1,\dots,a_n\} = \{b_1+c,\dots,b_n+c\} \text{ for some } c \in \mathbb{Z}.$$

7. Which of the following are rational functions on the named scrolls?

(1) x_1 on $\mathbb{F}(0)$.
(2) t_1x_2/x_1, t_1x_1/x_2 and t_2x_1/x_2 on $\mathbb{F}(1,0)$.
(3) $(x_1^2+x_2)/t_1x_3$ on $\mathbb{F}(1,2,3)$.
(4) $(x_1^3+t_1t_2x_1x_2^2)/(t_2x_1^2x_2+t_1^9x_3^3)$ on $\mathbb{F}(1,2,4)$.

Decide which of the following are bihomogeneous polynomials on $\mathbb{F}(0,3,5)$:

$$x_1^2+x_2, \quad t_1^3x_2+x_1^2, \quad t_1^3x_2+x_1, \quad x_3, \quad t_1t_2x_3+x_2.$$

8. Just as for projective space, a nonzero bihomogeneous polynomial g of bidegree $(d,e) \neq (0,0)$ is not a well-defined function on a scroll $\mathbb{F}$. Prove that for $P \in \mathbb{F}$ the condition $g(P) = 0$ is well defined, so that g defines a hypersurface in $\mathbb{F}$. [Hint: If you don't see this, prove it by lifting $P \in \mathbb{F}$ to different representatives $\widetilde{P} \in (\mathbb{C}^2 - 0) \times (\mathbb{C}^n - 0)$, and evaluating g at these points.]

9. Convince yourself that any two curves of the same degree in $\mathbb{P}^2$ are linearly equivalent. Now prove the same for any two hypersurfaces $X_{d,e} \subset \mathbb{F}$ of the same bidegree on a scroll $\mathbb{F}$. [Hint: Because the ratio of their equations is a rational function.]

Let $\mathbb{F}_a = \mathbb{F}(0,a)$ be the surface scroll, and D_1, D_2 the sections defined by $(x_1 = 0)$ and $(x_2 = 0)$. Find all divisors linearly equivalent to D_1 and containing D_2.

10. Let $Q_3 \subset \mathbb{P}^3$ be a quadric of rank 3 and $\mathbb{F}_2 \to Q_3$ its natural resolution (see 2.6). Study curves in Q_3 in terms of $\mathbb{F}_2$.

11. (a) Using bihomogeneous polynomials of bidegree $(2,-4)$ on $\mathbb{F}(1,2,3)$, write down a nonsingular surface $X_{2,-4} \subset \mathbb{F}(1,2,3)$ of bidegree $(2,-4)$. How many singular fibres does the conic bundle $X_{2,-4} \to \mathbb{P}^1$ have?

(b) The same question for $X_{2,-3} \subset \mathbb{F}(1,2,3)$.

(c) Describe (in terms of its fibres) the locus $(t_1^2 x_2^3 + t_1 t_2 x_1^2 x_3 = 0) \subset \mathbb{F}(1,2,4)$.

12. Let $X = X_{2,e} \subset \mathbb{F}(a_1, a_2, a_3)$ be a surface of bidegree $(2,e)$. The fibres of $X \to \mathbb{P}^1$ are plane conics. Prove that, if X is nonsingular, then every fibre is either a nonsingular conic or line pair. [Hint: You have to show that a double line leads to a singularity of X.]

Deduce a formula for the number of line pairs. [Hint: Singular conics are detected by a determinant, and you have to find its degree in t_1, t_2. Compare [UAG], proof of Proposition 7.3.]

13. Consider $\mathbb{F}(a_1, \dots, a_n)$ with some of the $a_i < 0$. When is the rational map of Theorem 2.5 defined? When is it the constant map? What is the dimension of its image? When is it in fact a morphism? Compare this with Theorem 2.8.

14. Prove the statement on linear generation of scrolls given in Remark 2.5, (a). [Hint: Write down the equation of all the rational normal curves of degree a_i, then the condition that corresponding points are joined up. Compare with the equations in Theorem 2.5.]

15. Generalise Theorem 2.5 to the case that some of the $a_i = 0$; compare Remark 2.5, (b).

16. Let $\Pi = \mathbb{P}^{N-2} : (x_0 = x_1 = 0) \subset \mathbb{P}^N$ be a codimension 2 linear subspace, and let $Q_1, \dots, Q_{N-n}$ be linearly independent quadrics containing Π. Prove that the intersection $\bigcap Q_i$ consists of Π together with an n-dimensional variety $\overline{\mathbb{F}}$ that is the image of a scroll under a linear embedding. [Hint: The ratio $(x_0 : x_1)$ defines a rational map $\mathbb{P}^N \to \mathbb{P}^1$, whose fibres are the pencil of hyperplanes through Π. Fibre-by-fibre, each Q_i defines a hyperplane. If the statement is true for some c then Q_{c+1} is a divisor in the scroll for c.]

17. Let $\mathbb{F}_a = \mathbb{F}(a,0)$ be the surface scroll as in 2.9, and consider the linear system $|D| = |\alpha A + 4B|$ for suitable $\alpha \in \mathbb{Z}$. Prove that $|D|$ is very ample for $\alpha > 4a$, and find out what happens when $\alpha = 4a$. Use the Newton polygon argument to prove that for $3a \le \alpha < 4a$, the general element of $|\alpha A + 4B|$ is of the form $B + C$ where B is the negative section and C is a nonsingular curve having $\alpha - 3a$ transverse points of intersection with B.

If α is even, study the elliptic surface obtained as double cover of $\mathbb{F}_a$ branched in a general element of $|\alpha A + 4B|$.

18. Suppose that $a_1 < \cdots < a_n$. Show that an automorphism of $\mathbb{F}(a_1, \dots, a_n)$ compatible with the projection $\mathbb{F} \to \mathbb{P}^1$ is of the form

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where $M = \{m_{ij}\}$ is an uppertriangular matrix with entries $m_{ij}(t_1, t_2)$ homogeneous polynomials of degree $a_j - a_i$.

What happens if $a_1 \le \cdots \le a_n$, with some equalities allowed?

19. Prove del Pezzo's theorem: an irreducible surface spanning $\mathbb{P}^n$ and of degree $n-1$ is either a scroll $\mathbb{F}(a_1, a_2)$ with $a_1 + a_2 = n-1$, or a cone $\overline{\mathbb{F}}(n-1, 0)$, or $\mathbb{P}^2$ if $n = 2$ or the Veronese surface if $n = 5$.

20. Suppose that $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_m$. Prove that there exists a surjective homomorphism

$$\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n) \twoheadrightarrow \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m)$$

if and only if $m \le n$ and for every i,

$$a_i \le b_i, \quad \text{and if } (a_1, \dots, a_i) \ne (b_1, \dots, b_i) \text{ then also } b_{i+1} \le a_i.$$

If $0 < a_1$, deduce necessary and sufficient conditions for $\mathbb{F}(b_1, \dots, b_{n-1})$ to be a hyperplane section of $\mathbb{F}(a_1, \dots, a_n)$.

21. Problem: find necessary and sufficient conditions for the existence of a short exact sequence

$$0 \to \mathcal{O}(c_1) \oplus \cdots \oplus \mathcal{O}(c_{n-m}) \to \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n) \to \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \to 0.$$

22. If $a_1 \le a_2$ and $a_1' \le a_2'$, prove that $\mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$ has a small deformation isomorphic to $\mathcal{O}(a_1') \oplus \mathcal{O}(a_2')$ if and only if $a_1 + a_2 = a_1' + a_2'$ and $a_1 \le a_1' \le a_2' \le a_1$. [Hint: You can find small deformations of $\mathbb{F}(a_1, a_2)$ by taking it as a "special" hyperplane section of a 3-fold $\mathbb{F}(b_1, b_2, b_3)$, then varying the hyperplane.]

23. Problem: find necessary and sufficient conditions for $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$ to have a small deformation isomorphic to $\mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m)$.

24. By analogy with the relative cubics of 2.10, consider the scroll $\mathbb{F}(a_1, a_2, a_3)$ with $0 \le a_1 \le a_2 \le a_3$, and relative quartic surfaces $X \in |(2 - \sum a_i)A + 4M|$. (These are the surfaces of general type on the Castelnuovo–Horikawa line $K^2 = 3p_g - 7$.)

25. By analogy with the elliptic surface of Ex. 2.17 obtained as double cover of the surface scroll $\mathbb{F}_a = \mathbb{F}(a, 0)$ branched in a curve $|D| = |2\alpha A + 4B|$, study the linear system $|D| = |2\alpha A + 6B|$ on $\mathbb{F}_a$ and the double cover branched in $|D| = |2\alpha A + 6B|$. (These are the surfaces of general type on the Max Noether–Horikawa line $K^2 = 2p_g - 4$.)