Rational curves on manifolds and Mori theory
BMS Course, Freie Universität Berlin Fall 2011

Problem set IV, for November 8th

The first few exercises concern topics discussed in the latest lecture. In the last one we deal with Kähler differentials which are defined in Hartshorne's book, section II.8. We recall the working definition. Let A be a finitely generated k algebra. For simplicity assume it is a domain (but this is not necessary). The A module Ω_A is generated by symbols da, with $a \in A$ modulo the relations: (1) $d(a+b) = da + db$, (2) $d(ab) = adb + bda$, (3) $d_{|k} = 0$. Clearly we have a k-linear map $d : A \to \Omega_A$ and any k-linear map $d' : A \to M$ to an A-module M which satisfies the above conditions factors through d. On a variety X we define the coherent sheaf Ω_X by glueing its values on the affine pieces.

1. Representation of $\mathcal{O}(1)$ as a Čech cocycle. This is to clarify possible confusions.[0] Verify from the definition that the cocycle $(g_{ij} = x_i/x_j) \in H^1((U_i), \mathcal{O}^*)$ defines the divisor (line bundle, invertible sheaf) $\mathcal{O}(1)$ on $\mathbb{P}^n$ with homogeneous coordinates $[x_0, \ldots, x_n]$ and standard covering (U_i). Verify that for $r = 0, \ldots, n$ functions x_r/x_i on U_i glue to global sections of the line bundle (or invertible sheaf) $\mathcal{O}(1)$. Note that the choice of a divisor, e.g. $H_0 = V(x_0)$ determines an isomorphism of $\mathcal{O}(1)$ with $\mathcal{O}(H_0)$ where the latter sheaf is treated as a subsheaf of the constant sheaf $k(\mathbb{P}^n)$.

2. Let $S = \bigoplus_{d \geq 0} S^d = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. Recall that for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ we associate a graded S-module of its sections $\Gamma^*(\mathcal{F}) = \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j))$. Here $\mathcal{F}(j) = \mathcal{F} \otimes \mathcal{O}(j)$.

   (a) Verify that $\Gamma^*(\mathcal{O}) = S$

   (b) Verify that $\Gamma^*(\mathcal{F}(r)) = \Gamma^*(\mathcal{F})(r)$, where for a graded module $M = \bigoplus_j M_j$ the module $M(r)$ satisfies $M(r)_j = M_{r+j}$.

3. Recall that for a graded S-module $M = \bigoplus_{j \in \mathbb{Z}} M^j$ we define a quasi-coherent sheaf $\widetilde{M}$ on $\mathbb{P}^n$ such that on every affine set U_i we have

[0] The sheaf $\mathcal{O}(d)$ was defined in exercise I.6 and then redefined during the lecture as the sheaf associated to module $S(d)$; I want to make clear that these objects are the same (up to isomorphism).

$\widetilde{M}(U_i) = M^0_{x_i} = \{m/x_i^r : m \in M^r\}$. Given $\mathcal{F}$, a quasi-coherent sheaf over $\mathbb{P}^n$, we define a morphism of sheaves $\widetilde{\Gamma^*(\mathcal{F})} \longrightarrow \mathcal{F}$ which for $s/x_i^j \in \widetilde{\Gamma^*(\mathcal{F})}(U_i)$, with $s \in H^0(\mathbb{P}^n, \mathcal{F}(j))$, associates $s \otimes x_i^{-j} \in \mathcal{F}(U_i)$. Prove that this is isomorphism hence $\widetilde{\Gamma^*(\mathcal{F})} \simeq \mathcal{F}$

4. Comparing $\Gamma^*(\widetilde{M})$ with M.

   (a) Let $I^r = \bigoplus_{d \geq r} S^d$. Find the graded module $\Gamma^*(\widetilde{I^d})$.

   (b) [a bit harder] Let M be a graded S module. Show that there exists a natural homomorphism $M \longrightarrow \Gamma^*(\widetilde{M})$ which, if M is finitely generated, is isomorphism for high degrees. That is, $M^d \simeq \Gamma^*(\widetilde{M})^d$ for $d \gg 0$.

5. Let $\phi : M \longrightarrow N$ be a homomorphism of graded finitely generated S modules. Prove that it gives rise to a homomorphism $\widetilde{\phi} : \widetilde{M} \longrightarrow \widetilde{N}$ of coherent sheaves. Prove that if ϕ is injective then $\widetilde{\phi}$ is injective. Prove that if ϕ is surjective for sufficiently large degrees than $\widetilde{\phi}$ is surjective. Can one inverse the latter two statements?

6. Euler sequence [Hartshorne, II.8.13]. Consider a morphism of graded S modules

$$\bigoplus_{j=0}^{n} S(-1) \cdot v_j \longrightarrow S$$

   where v_j's are generators of degree 1 and each v_j is mapped to x_j. Let M be the kernel of this map.

   (a) Show that we get the exact sequence of $\mathcal{O}$ sheaves

$$0 \longrightarrow \widetilde{M} \longrightarrow \mathcal{O}(-1)^{\oplus n+1} \longrightarrow \mathcal{O} \longrightarrow 0$$

   (b) On U_i with coordinate ring $k[x_0/x_i, \ldots, x_n/x_i]$ we define a map $\psi_i : \Omega_{U_i} \to \mathcal{O}(-1)^{\oplus n+1}_{|U_i}$ by setting $\psi_i(d(x_r/x_i)) = (x_i v_r - x_r v_i)/x_i^2$. Prove that the map is isomorphism onto $\widetilde{M}_{|U_i}$.

   (c) Prove that the homomorphisms ψ_i glue to isomorphism of sheaves $\Omega_{\mathbb{P}^n} \to \widetilde{M}$ so that we have the Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}(-1)^{\oplus n+1} \longrightarrow \mathcal{O} \longrightarrow 0$$