Rational curves on manifolds and Mori theory
BMS Course, Freie Universität Berlin Fall 2011

Problem set I, for October 18th
Review on Čech cohomology.

Suppose that X is a topological space. In what follows, will usually assume X to be a variety over an algebraically closed field k (always algebraically closed!).
We choose $\mathcal{U} = (U_i)_{1=1}^r$ a finite open covering of X. For any set of indexes $I \subset \{1, \dots, r\}$ of cardinality $1 \leq |I| \leq r$ we set $U_I = \bigcup_{i \in I} U_i$.
For $\mathcal{U}$ as above and $\mathcal{F}$, a sheaf of abelian groups over X we define the Čech complex

$$0 \longrightarrow \prod_{|I|=1} \mathcal{F}(U_I) \longrightarrow \prod_{|I|=2} \mathcal{F}(U_I) \longrightarrow \prod_{|I|=3} \mathcal{F}(U_I) \longrightarrow \cdots$$

with differentiation map $\delta^p : \prod_{|I|=p} \mathcal{F}(U_I) \longrightarrow \prod_{|I|=p+1} \mathcal{F}(U_I)$ which to $\sigma = (\sigma_{i_1 < \cdots < i_p}) \in \prod_{|I|=p} \mathcal{F}(U_I)$ associates $\sigma' = (\sigma'_{i_0 < \cdots < i_p}) \in \prod_{|I|=p+1} \mathcal{F}(U_I)$ such that

$$\sigma'_{i_0 < \cdots < i_p} = \sum_{j=0}^{p} (-1)^j \sigma_{i_0 < \cdots \widehat{i_j} \cdots < i_p}$$

where $\widehat{i_j}$ means omission of the index and the right-hand summands are restricted to $U_{i_0} \cap \cdots \cap U_{i_p}$. We define the p-th Čech cohomology as the cohomology of the above complex, that is

$$H^p(\mathcal{U}, \mathcal{F}) = \ker \delta^{p+1} / \operatorname{im} \delta^p$$

For an inscribed covering $\mathcal{U}' \preceq \mathcal{U}$ we have the restriction map $H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{U}', \mathcal{F})$ and the Čech cohomology is defined as the direct (injective) limit. We denote it $\check{H}^p(X, \mathcal{F})$ or just $H^p(X, \mathcal{F})$
As a consequence of Leray theorem if $\mathcal{F}$ is a quasicoherent sheaf and $\mathcal{U}$ is an affine then, actually, $H^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(X, \mathcal{F})$ is an isomorphism.
Notation, convention: the elements of $\prod_{|I|=p+1} \mathcal{F}(U_I)$ are called cochains, the elements in $\ker \delta$ cocycles and in $\operatorname{im} \delta$ coboundaries. Sometimes it is convenient to write the set of indexes I not in the increasing order: if we change the order then we have to change the sign of σ, that is $\sigma_{i_1 \cdots i_p} = \operatorname{sgn}(\pi) \cdot \sigma_{\pi(i_1) \cdots \pi(i_p)}$

1. Check that the Čech complex is a complex, that is $\delta^{p+1} \circ \delta^p = 0$.

2. Using the Čech complex show the following general nonsense properties of Čech cohomology

   (a) $H^0(X, \mathcal{F}) = \mathcal{F}(X)$

   (b) $H^p(X, \mathcal{F})$ is a $H^0(X, \mathcal{F})$ module

   (c) $H^p(X, \ \cdot\ )$ is a covariant functor

   (d) For an exact sequence of sheaves of abelian groups

   $$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

   we have an exact sequence of groups

   $$0 \longrightarrow H^0(X, \mathcal{A}) \longrightarrow H^0(X, \mathcal{B}) \longrightarrow H^0(X, \mathcal{C}) \longrightarrow H^1(X, \mathcal{A}) \longrightarrow$$

   $$H^1(X, \mathcal{B}) \longrightarrow H^1(X, \mathcal{C}) \longrightarrow \cdots\cdots$$

   where the connecting homomorphism $H^p(X, \mathcal{C}) \to H^{p+1}(X, \mathcal{A})$ comes from snake-lemma construction used to a diagram consisting of three Čech complexes and the maps coming from maps of the sheaves which you may assume make short exact sequences. Checking exactness is tedious work but the idea is simple: divide the diagram to maps of two consecutive exact sequences and use snake lemma to produce connecting homomorphism for cohomology.

3. Acyclic resolution. Suppose that we have a long exact sequence of sheaves of abelian groups on a topological space X

   $$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B}^0 \longrightarrow \mathcal{B}^1 \longrightarrow \mathcal{B}^2 \longrightarrow \cdots \longrightarrow \mathcal{B}^m \longrightarrow 0$$

   and assume that all non-zero cohomology of sheaves $\mathcal{B}^i$ vanish (such $\mathcal{B}^i$ is called acyclic). Divide the above sequence into short sequences and prove that cohomology of $\mathcal{A}$ is equal to cohomology (kernel divided by the image of the respective arrow) of the following complex of global sections of the above sequence

   $$0 \longrightarrow \mathcal{B}^0(X) \longrightarrow \mathcal{B}^1(X) \longrightarrow \mathcal{B}^2(X) \longrightarrow \cdots \longrightarrow \mathcal{B}^m(X) \longrightarrow 0$$

4. A sheaf of rings $\mathcal{A}$ over a topological space admits a partition of unity if for every finite (for simplicity) covering (U_i) there exist sections $f_i \in \mathcal{A}(X)$, each f_i having support inside U_i (which means that there exists a closed set $K_i \subset U_i$ such that $f_{i|V} = 0$ for every $V \subset X \setminus K_i$) such that $\sum_i f_i = 1$. A sheaf $\mathcal{F}$ of $\mathcal{A}$ modules is called fine; below, we assume $\mathcal{F}$ is fine.

   (a) Let $\sigma \in \prod_{|I|=p+1} \mathcal{F}(U_I)$ be in the kernel of δ^{p+1}. We define $\widehat{\sigma} \in \prod_{|I|=p} \mathcal{F}(U_I)$ by the formula (think why does it make sense)

   $$\widehat{\sigma}_{i_1 \cdots i_p} = \sum_i f_i \sigma_{i, i_1 \cdots i_p}$$

   Prove that $\delta^p(\widehat{\sigma}) = \sigma$. (Note that our convention involves sign change.)

   (b) Show that $H^p(X, \mathcal{F}) = 0$ for $p > 0$.

   (c) If X is a differentiable manifold use differentials forms to construct an acyclic resolution of a locally constant sheaf $\mathbb{Z}_X$ and prove that its Čech cohomology is equal to de Rham cohomology of X: $H^p(X, \mathbb{R}_X) = H^r_{DR}(X)$.

5. Let $U_i = \{x_i \neq 0\}$ be an affine covering of $\mathbb{A}^2_k \setminus \{(0,0)\}$. Find $H^0((U_i), \mathcal{O})$, where $\mathcal{O}$ denotes the structural sheaf, and show that $H^1((U_i), \mathcal{O}) \neq 0$.

6. Let $\mathbb{P}^n_k$ be the projective space with homogeneous coordinates $[x_0, \ldots, x_n]$ and standard affine covering consisting of $U_i = \{x_i \neq 0\}$. Recall that the field of rational functions $k(\mathbb{P}^n_k)$ can be identified with homogeneous rational functions $k(x_0, \ldots, x_n)$ of degree 0 and $k[U_i] \subset k(\mathbb{P}^n_k)$, the functions regular on U_i, are of the form f/x_i^d where f is homogeneous of degree d in $k[x_0, \ldots, x_n]$. We define a sheaf $\mathcal{O}(d)$ as follows

   $$\mathcal{O}(d)(U_i) = \{f \in k(\mathbb{P}^n_k) : (x_0/x_i)^d \cdot f \in \mathcal{O}(U_i)\}$$

   where, of course, $\mathcal{O}(U)$ are functions regular on U.

   (a) Calculate $H^0((U_i), \mathcal{O}(d))$ and $H^1((U_i), \mathcal{O}(d))$ for $\mathbb{P}^1_k$ directly from the Čech complex.

(b) Calculate $H^p((U_i), \mathcal{O})$ for arbitrary $\mathbb{P}^n_k$, start with $n = 2$, for simplicity.

(c) Show that $\mathcal{O}(-1) \hookrightarrow \mathcal{O}$ is a sheaf of ideals of a hyperplane. Conclude the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow 0$$

(d) Use the above sequence and the information about $H^p(\mathbb{P}^n, \mathcal{O})$ to calculate $H^p(\mathbb{P}^n, \mathcal{O}(d))$ for arbitrary d.

More problems, other important properties of Čech cohomology.

A If X is affine variety (or scheme) and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}$ modules then $\mathcal{F}$ is acyclic, that is $H^p(X, \mathcal{F}) = 0$ for $p > 0$. (Hence we can use affine coverings to compute cohomology.) See e.g. [Ravi Vakil's book], Chapter 20.

B The above statement has its inverse: If X is a scheme such that every coherent sheaf on X (or even: every sheaf of ideals) is acyclic then X is affine. See e.g. [Ueno, exercise 6.1].

C Cohomology over an affine scheme may be nontrivial for non-coherent sheaves. Let $D \subset \mathbb{P}^2_k$ be an irreducible curve of degree $d > 0$. Set $U = \mathbb{P}^2_k \setminus D$. Then U is affine and $\operatorname{Pic} U = H^1(U, \mathcal{O}^*) = \mathbb{Z}/d\mathbb{Z}$, see [Hartshorne, II.6.5, II.6.11 and Ex. III.4.5]. Find more example (e.g. more sheaves on affine curves).