FANO VARIETIES. PROBLEM LIST.

General properties of Fano varieties

(1) Let $f : X \to Z$ be a projective surjective morphism of smooth varieties. Assume that X is Fano. Show that
  (a) $H^i(Z, \mathscr{O}_Z) = 0\ \forall i \geq 1$;
  (b) $H^0(Z, nK_Z) = 0\ \forall n \geq 1$;
  (c) if Z is a curve, then $Z \simeq \mathbb{P}^1$;
  (d) if Z is a surface, then Z is rational.

(2) Let X be a Fano variety. Show that $\pi_1^{\text{alg}}(X) = \{1\}$.

(3) Let X be a Fano variety and let $\varphi : X \to X$ be an automorphism of prime order. Show that φ has a fixed point.

(4) Let $X = \text{Gr}(n, k)$ be the Grassmann variety. Show that $\text{Pic}(X) \simeq \mathbb{Z}$. Compute the canonical class. Show that X is Fano.

(5) Let X be a smooth projective variety of dimension n and let $D \subset X$ be a prime divisor such that $X \setminus D \simeq \mathbb{A}^n$. Show that X is Fano. Give some examples.

(6) Let $\mathbb{P} := \mathbb{P}(a_0, \ldots, a_n)$ be a weighted projective space. Assume that the collection of weights $(a_0, \ldots, a_n)$ is *well-formed*, that is, $\gcd(a_0, \ldots, \hat{a_i} \ldots, a_n) = 1$ for all i. Prove that $K_{\mathbb{P}} = -(\sum a_i)A$, where A is the positive generator of $\text{Cl}(\mathbb{P})$, the Weil divisor class group.

(7) Let $f : Y \to X$ be the blowup of some number of points on a smooth projective variety. Assume that Y is a Fano. Show that so X is.

(8) Let $f : X \to \mathbb{P}^n$ be the blowup of a point. Show that X is Fano. What is the type of the second extremal ray? Can the blowup of two points on $\mathbb{P}^n$ be a Fano variety?

(9) Let X be a three-dimensional variety and let $f : X \to \mathbb{P}^1$ be a *smooth* projective morphism such that $-K_X$ is relatively ample. Show that $\rho(X) = 11 - K_{X_\eta}^2$.

Del Pezzo surfaces

(10) Let $X \subset \mathbb{P}^n$ be a smooth surface such that its general hyperplane section is an elliptic curve. Show that X is either a del Pezzo surface or a $\mathbb{P}^1$-bundle over an elliptic curve..

(11) Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a smooth divisor of bidegree $(1, 1)$. Show that X is a del Pezzo surface. Compute the degree of X.

(12) Show that any del Pezzo surface of degree 6 is isomorphic to a smooth divisor of tridegree $(1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Write down such a surface $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by an explicit equation and find all the lines on S.

(13) Show that any del Pezzo surface of degree 6 is isomorphic to a smooth intersection of two divisors of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

(14) Find all del Pezzo surfaces that are isomorphic to complete intersections of hypersurfaces in some Grassmannian $\mathrm{Gr}(k, N) \subset \mathbb{P}(\wedge^k \mathbb{C}^n)$.
(15) Show that any del Pezzo surface of degree 5 is isomorphic to a smooth section of $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ by a subspace of codimension 4.
(16) Show that any del Pezzo surface of degree 5 is isomorphic to a smooth divisor of bidegree $(1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Show that the projection to $\mathbb{P}^2$ is the blowup of 4 points.
(17) Show that any del Pezzo surface of degree 4 is isomorphic to an intersection of two qudrics in $\mathbb{P}^4$.
(18) Let $Q_1, Q_2 \subset \mathbb{P}^4$ be two distinct quadrics. Show that the intersection $Q_1 \cap Q_2$ is smooth if and only if the pencil $\langle Q_1, Q_2 \rangle$ contains exactly 5 degenerate members.
(19) Show that any del Pezzo surface of degree 4 can be given in $\mathbb{P}^4$ by the equations

$$x_0^2 + \cdots + x_4^2 = \lambda_0 x_0^2 + \cdots + \lambda_4 x_4^2 = 0.$$

(20) Let $X \subset \mathbb{P}^4$ be a del Pezzo surface of degree 4 and let $\mathcal{Q}$ be the pencil of quadrics in $\mathbb{P}^4$ passing through X. Show that 5 degenerate quadrics $Q_i \in \mathcal{Q}$ define 5 double coverings $X \to \mathbb{P}^1 \times \mathbb{P}^1$. Therefore, there are 5 (biregular) involutions τ_i on X. What are branch divisors of these coverings? Write down involutions τ_i explicitly and show that they generate a normal subgroup $N \subset \mathrm{Aut}(X)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.
(21) Let $X \subset \mathbb{P}^3$ be a cubic given by $x_0^3 + \cdots + x_3^3 = 0$ (Fermat cubic). Find all the lines on X. Compute the automorphism group.
(22) Let $X \subset \mathbb{P}^3 \subset \mathbb{P}^4$ be a cubic given by $x_0 + \cdots + x_4 = x_0^3 + \cdots + x_4^3 = 0$ (Clebsch diagonal cubic). Find all the lines on X. Compute the automorphism group.
(23) A point P on a cubic surface $X \subset \mathbb{P}^3$ is called an *Eckardt point* if there are three lines $L_i \subset X$ passing through P. Show that a general cubic surface contains no Eckardt points. Compute the codimension of the family of all cubic surfaces having Eckardt points.
(24) Show that on a del Pezzo surface X of degree $2 \le d \le 7$ any effective divisor is linearly equivalent to a linear combination of lines with non-negative integer coefficients. Is it true for del Pezzos of degree 1?
(25) Let X be a del Pezzo surface and let $\tau \in \mathrm{Aut}(X)$ be an element of order 2 such that the locus of fixed points is a finite set. Prove that K_X^2 is even.
(26) Find all del Pezzo surfaces that are isomorphic to weighted complete intersections in some weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$.

Fano threefolds

(27) Find all Fano threefolds that are isomorphic to complete intersections of hypersurfaces in some Grassmannian $\mathrm{Gr}(k,N) \subset \mathbb{P}(\wedge^k \mathbb{C}^n)$.
(28) Find all Fano threefolds that are isomorphic to complete intersections of divisors in products $\mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m}$.
(29) Let X be a Fano threefold and let $f : X \to Z$ be a surjective morphism to a smooth surface Z. Show that Z is del Pezzo. *Hint*. Use the formula $-4K_Z \equiv f_* K_X^2 + \Delta$, where $\Delta \subset Z$ is the discriminant curve.
(30) Show that the complete flag variety of $\mathbb{P}^2$ is a Fano threefold $V_6 \subset \mathbb{P}^7$.
(31) Let $X \subset \mathbb{P}^4$ be a smooth cubic hypersurface and let $L \subset X$ be a line. Let $f : Y \to X$ be the blowup of L. Show that Y is Fano. What is the type of the second extremal ray?
(32) Let $X \subset \mathbb{P}^5$ be a smooth intersection of two quadrics and let $L \subset X$ be a line. Let $f : Y \to X$ be the blowup of L. Show that Y is Fano. What is the type of the second extremal ray?
(33) Let $f : X \to \mathbb{P}^3$ be the blowup of a smooth curve $C \subset \mathbb{P}^3$. Find a sufficient condition for X to be a Fano variety.
(34) Let $Q \subset \mathbb{P}^4$ be a smooth quadric and let $f : X \to Q$ be the blowup of a smooth curve $C \subset Q$. Find a sufficient condition for X to be a Fano variety.
(35) Let $V = V_d \subset \mathbb{P}^{d+1}$ be a del Pezzo threefold and let $f : X \to V$ be the blowup of a smooth curve $C \subset V$. Find a sufficient condition for X to be a Fano variety.
(36) Let $f : X \to \mathbb{P}^3$ be the blowup of a smooth curve $C \subset \mathbb{P}^3$. Find a sufficient condition for X to be a Fano variety.

Automorphisms of Fano varieties

(37) Show that a general del Pezzo surface of degree 3 has no non-trivial automorphisms.
(38) Show that the only non-trivial automorphism of a general del Pezzo surface of degree 1 (resp. 2) is the Bertini (resp. Geiser) involution.
(39) Describe the automorphism group of a del Pezzo surface of degree 6.
(40) Prove that the automorphism group of a del Pezzo surface of degree 5 is isomorphic to S_5.
(41) Describe the automorphism group of $V_6 \subset \mathbb{P}^7$.
(42) Prove that the automorphism group of $V_5 \subset \mathbb{P}^6$ is isomorphic to PSL_2.
(43) Using the action of PSL_2 on $V_5 \subset \mathbb{P}^6$ describe the family of lines.

(44) Let $C \subset \mathbb{P}^4$ be a rational normal curve of degree 4. The action of the group $\mathrm{Aut}(C) = \mathrm{PGL}_2(\Bbbk)$ naturally extends to $\mathbb{P}^4$. Show that there exists an invariant non-singular quadric $Q \subset \mathbb{P}^4$ containing C. Let $f_1 : X \to Q$ be the blowup of C. Show that X is a Fano threefold admitting a $\mathrm{PGL}_2(\Bbbk)$-action.

(45) Let $\Gamma \subset \mathbb{P}^2$ is a non-degenerate conic and let $\Gamma^* \subset \mathbb{P}^{2*}$ be its dual, the conic formed by lines that are tangent to Γ. Consider the incidence curve

$$C = \{(P, L) \in \Gamma \times \Gamma^* \subset \mathbb{P}^2 \times \mathbb{P}^{2*} \mid L \text{ is tangent to } \Gamma \text{ at } P\}.$$

Then C is contained into the flag variety $\mathrm{Fl}(\mathbb{P}^2) = V_6$. The action $\mathrm{Aut}(\Gamma) = \mathrm{PGL}_2(\Bbbk)$ extends to $V_6 = \mathrm{Fl}(\mathbb{P}^2)$. Let $f : X \to V_6$ be the blowup of C. Show that X is a Fano threefold admitting a $\mathrm{PGL}_2(\Bbbk)$-action.

(46) Let $(a_{i,j})$, $(b_{i,j})$, $(c_{i,j})$ be symmetric 4×4-matrices and let $X \subset \mathbb{P}^3_{x_1 \dots, x_4} \times \mathbb{P}^3_{y_1 \dots, y_4}$ is given by the equations

$$\sum a_{i,j} x_i y_j = \sum b_{i,j} x_i y_j = \sum c_{i,j} x_i y_j = 0.$$

If $(a_{i,j})$, $(b_{i,j})$, $(c_{i,j})$ are taken sufficiently general, then X is a smooth Fano threefold admitting an action of $\boldsymbol{\mu}_2$. The projections to both copies of $\mathbb{P}^3$ are blowups of curves of degree 6 and genus 3.

(47) Let X be a del Pezzo surface of degree 1 (resp. 2) and let $\tau \in \mathrm{Aut}(X)$ be the Bertini (resp. Geiser) involution. Show that
  (a) the pair (τ, X) is minimal;
  (b) τ is not linearizable;
  (c) τ is not conjugate to a de Jonquiéres involution;
  (d) if $\tau' \in \mathrm{Aut}\, X$ is an involution such that (τ', X) is minimal, then $\tau' = \tau$.