Fano varieties. Problem list.

General properties of Fano varieties

(1) Let \( f : X \to Z \) be a projective surjective morphism of smooth varieties. Assume that \( X \) is Fano. Show that
   (a) \( H^i(Z, \mathcal{O}_Z) = 0 \) \( \forall i \geq 1 \);
   (b) \( H^0(Z, nK_Z) = 0 \) \( \forall n \geq 1 \);
   (c) if \( Z \) is a curve, then \( Z \simeq \mathbb{P}^1 \);
   (d) if \( Z \) is a surface, then \( Z \) is rational.

(2) Let \( X \) be a Fano variety. Show that \( \pi_{\text{alg}}(X) = \{1\} \).

(3) Let \( X \) be a Fano variety and let \( \varphi : X \to X \) be an automorphism of prime order. Show that \( \varphi \) has a fixed point.

(4) Let \( X = \text{Gr}(n, k) \) be the Grassmann variety. Show that \( \text{Pic}(X) \simeq \mathbb{Z} \). Compute the canonical class. Show that \( X \) is Fano.

(5) Let \( X \) be a smooth projective variety of dimension \( n \) and let \( D \subset X \) be a prime divisor such that \( X \setminus D \simeq \mathbb{A}^n \). Show that \( X \) is Fano. Give some examples.

(6) Let \( \mathbb{P} := \mathbb{P}(a_0, \ldots, a_n) \) be a weighted projective space. Assume that the collection of weights \( (a_0, \ldots, a_n) \) is well-formed, that is, \( \gcd(a_0, \ldots, a_i, \ldots, a_n) = 1 \) for all \( i \). Prove that \( K_{\mathbb{P}} = -(\sum a_i)A \), where \( A \) is the positive generator of \( \text{Cl}(\mathbb{P}) \), the Weil divisor class group.

(7) Let \( f : Y \to X \) be the blowup of some number of points on a smooth projective variety. Assume that \( Y \) is a Fano. Show that so \( X \) is.

(8) Let \( f : X \to \mathbb{P}^n \) be the blowup of a point. Show that \( X \) is Fano. What is the type of the second extremal ray? Can the blowup of two points on \( \mathbb{P}^n \) be a Fano variety?

(9) Let \( X \) be a three-dimensional variety and let \( f : X \to \mathbb{P}^1 \) be a smooth projective morphism such that \( -K_X \) is relatively ample. Show that \( \rho(X) = 11 - K_X^3 \).

Del Pezzo surfaces

(10) Let \( X \subset \mathbb{P}^n \) be a smooth surface such that its general hyperplane section is an elliptic curve. Show that \( X \) is either a del Pezzo surface or a \( \mathbb{P}^1 \)-bundle over an elliptic curve.

(11) Let \( X \subset \mathbb{P}^1 \times \mathbb{P}^2 \) be a smooth divisor of bidegree \((1, 1)\). Show that \( X \) is a del Pezzo surface. Compute the degree of \( X \).

(12) Show that any del Pezzo surface of degree 6 is isomorphic to a smooth divisor of tridegree \((1, 1, 1)\) in \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). Write down such a surface \( S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) by an explicit equation and find all the lines on \( S \).

(13) Show that any del Pezzo surface of degree 6 is isomorphic to a smooth intersection of two divisors of bidegree \((1, 1)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \).
(14) Find all del Pezzo surfaces that are isomorphic to complete intersections of hypersurfaces in some Grassmannian $\text{Gr}(k, N) \subset \mathbb{P}(\wedge^k \mathbb{C}^n)$.

(15) Show that any del Pezzo surface of degree 5 is isomorphic to a smooth section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$ by a subspace of codimension 4.

(16) Show that any del Pezzo surface of degree 5 is isomorphic to a smooth divisor of bidegree $(1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$. Show that the projection to $\mathbb{P}^2$ is the blowup of 4 points.

(17) Show that any del Pezzo surface of degree 4 is isomorphic to an intersection of two quadrics in $\mathbb{P}^4$.

(18) Let $Q_1, Q_2 \subset \mathbb{P}^4$ be two distinct quadrics. Show that the intersection $Q_1 \cap Q_2$ is smooth if and only if the pencil $\langle Q_1, Q_2 \rangle$ contains exactly 5 degenerate members.

(19) Show that any del Pezzo surface of degree 4 can be given in $\mathbb{P}^4$ by the equations
\[ x_0^2 + \cdots + x_4^2 = \lambda_0 x_0^2 + \cdots + \lambda_4 x_4^2 = 0. \]

(20) Let $X \subset \mathbb{P}^4$ be a del Pezzo surface of degree 4 and let $\mathcal{Q}$ be the pencil of quadrics in $\mathbb{P}^4$ passing through $X$. Show that 5 degenerate quadrics $Q_i \in \mathcal{Q}$ define 5 double coverings $X \to \mathbb{P}^1 \times \mathbb{P}^1$. Therefore, there are 5 (biregular) involutions $\tau_i$ on $X$. What are branch divisors of these coverings? Write down involutions $\tau_i$ explicitly and show that they generate a normal subgroup $N \subset \text{Aut}(X)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.

(21) Let $X \subset \mathbb{P}^3$ be a cubic given by $x_0^3 + \cdots + x_3^3 = 0$ (Fermat cubic). Find all the lines on $X$. Compute the automorphism group.

(22) Let $X \subset \mathbb{P}^3 \subset \mathbb{P}^4$ be a cubic given by $x_0 + \cdots + x_4 = x_0^3 + \cdots + x_4^3 = 0$ (Clebsch diagonal cubic). Find all the lines on $X$. Compute the automorphism group.

(23) A point $P$ on a cubic surface $X \subset \mathbb{P}^3$ is called an Eckardt point if there are three lines $L_i \subset X$ passing through $P$. Show that a general cubic surface contains no Eckardt points. Compute the codimension of the family of all cubic surfaces having Eckardt points.

(24) Show that on a del Pezzo surface $X$ of degree $2 \leq d \leq 7$ any effective divisor is linearly equivalent to a linear combination of lines with non-negative integer coefficients. Is it true for del Pezzos of degree 1?

(25) Let $X$ be a del Pezzo surface and let $\tau \in \text{Aut}(X)$ be an element of order 2 such that the locus of fixed points is a finite set. Prove that $K_X^2$ is even.

(26) Find all del Pezzo surfaces that are isomorphic to weighted complete intersections in some weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$.
Fano threefolds

(27) Find all Fano threefolds that are isomorphic to complete intersections of hypersurfaces in some Grassmannian \( \text{Gr}(k, N) \subset \mathbb{P}(\wedge^k \mathbb{C}^n) \).

(28) Find all Fano threefolds that are isomorphic to complete intersections of divisors in products \( \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_m} \).

(29) Let \( X \) be a Fano threefold and let \( f : X \to Z \) be a surjective morphism to a smooth surface \( Z \). Show that \( Z \) is del Pezzo. *Hint.* Use the formula \(-4K_Z \equiv f_* K_X^2 + \Delta \), where \( \Delta \subset Z \) is the discriminant curve.

(30) Show that the complete flag variety of \( \mathbb{P}^2 \) is a Fano threefold \( V_6 \subset \mathbb{P}^7 \).

(31) Let \( X \subset \mathbb{P}^4 \) be a smooth cubic hypersurface and let \( L \subset X \) be a line. Let \( f : Y \to X \) be the blowup of \( L \). Show that \( Y \) is Fano. What is the type of the second extremal ray?

(32) Let \( X \subset \mathbb{P}^5 \) be a smooth intersection of two quadrics and let \( L \subset X \) be a line. Let \( f : Y \to X \) be the blowup of \( L \). Show that \( Y \) is Fano. What is the type of the second extremal ray?

(33) Let \( f : X \to \mathbb{P}^3 \) be the blowup of a smooth curve \( C \subset \mathbb{P}^3 \). Find a sufficient condition for \( X \) to be a Fano variety.

(34) Let \( Q \subset \mathbb{P}^4 \) be a smooth quadric and let \( f : X \to Q \) be the blowup of a smooth curve \( C \subset Q \). Find a sufficient condition for \( X \) to be a Fano variety.

(35) Let \( V = V_d \subset \mathbb{P}^{d+1} \) be a del Pezzo threefold and let \( f : X \to V \) be the blowup of a smooth curve \( C \subset V \). Find a sufficient condition for \( X \) to be a Fano variety.

(36) Let \( f : X \to \mathbb{P}^3 \) be the blowup of a smooth curve \( C \subset \mathbb{P}^3 \). Find a sufficient condition for \( X \) to be a Fano variety.

**Automorphisms of Fano varieties**

(37) Show that a general del Pezzo surface of degree 3 has no non-trivial automorphisms.

(38) Show that the only non-trivial automorphism of a general del Pezzo surface of degree 1 (resp. 2) is the Bertini (resp. Geiser) involution.

(39) Describe the automorphism group of a del Pezzo surface of degree 6.

(40) Prove that the automorphism group of a del Pezzo surface of degree 5 is isomorphic to \( S_5 \).

(41) Describe the automorphism group of \( V_6 \subset \mathbb{P}^7 \).

(42) Prove that the automorphism group of \( V_5 \subset \mathbb{P}^6 \) is isomorphic to \( PSL_2 \).

(43) Using the action of \( PSL_2 \) on \( V_5 \subset \mathbb{P}^6 \) describe the family of lines.
(44) Let \( C \subset \mathbb{P}^4 \) be a rational normal curve of degree 4. The action of the group \( \text{Aut}(C) = \text{PGL}_2(\mathbb{k}) \) naturally extends to \( \mathbb{P}^4 \). Show that there exists an invariant non-singular quadric \( Q \subset \mathbb{P}^4 \) containing \( C \). Let \( f_1 : X \to Q \) be the blowup of \( C \). Show that \( X \) is a Fano threefold admitting a \( \text{PGL}_2(\mathbb{k}) \)-action.

(45) Let \( \Gamma \subset \mathbb{P}^2 \) is a non-degenerate conic and let \( \Gamma^* \subset \mathbb{P}^2^* \) be its dual, the conic formed by lines that are tangent to \( \Gamma \). Consider the incidence curve
\[
C = \{(P, L) \in \Gamma \times \Gamma^* \subset \mathbb{P}^2 \times \mathbb{P}^2^* \mid L \text{ is tangent to } \Gamma \text{ at } P \}.
\]
Then \( C \) is contained into the flag variety \( \text{Fl}(\mathbb{P}^2) = V_6 \). The action \( \text{Aut}(\Gamma) = \text{PGL}_2(\mathbb{k}) \) extends to \( V_6 = \text{Fl}(\mathbb{P}^2) \). Let \( f : X \to V_6 \) be the blowup of \( C \). Show that \( X \) is a Fano threefold admitting a \( \text{PGL}_2(\mathbb{k}) \)-action.

(46) Let \( (a_{i,j}), (b_{i,j}), (c_{i,j}) \) be symmetric \( 4 \times 4 \)-matrices and let \( X \subset \mathbb{P}^3 \times \mathbb{P}^3 \) be given by the equations
\[
\sum a_{i,j}x_iy_j = \sum b_{i,j}x_iy_j = \sum c_{i,j}x_iy_j = 0.
\]
If \( (a_{i,j}), (b_{i,j}), (c_{i,j}) \) are taken sufficiently general, then \( X \) is a smooth Fano threefold admitting an action of \( \mu_2 \). The projections to both copies of \( \mathbb{P}^3 \) are blowups of curves of degree 6 and genus 3.

(47) Let \( X \) be a del Pezzo surface of degree 1 (resp. 2) and let \( \tau \in \text{Aut}(X) \) be the Bertini (resp. Geiser) involution. Show that
(a) the pair \((\tau, X)\) is minimal;
(b) \( \tau \) is not linearizable;
(c) \( \tau \) is not conjugate to a de Jonquières involution;
(d) if \( \tau' \in \text{Aut} X \) is an involution such that \((\tau', X)\) is minimal, then \( \tau' = \tau \).