FINITE SUBGROUPS OF THE CREMONA GROUP OF THE PLANE - EXERCISES

1. Exercises that can be tried at any time

Exercise 1.1. *For any polynomial* $p \in \mathbf{k}(x)$, *let* φ_p *be the birational map of* $\mathbb{A}^2$ *given by* $(x,y) \dashrightarrow (x, \frac{p}{y})$. *(These are the classical* de Jonquières involutions*). Decide for which* p, p' *the maps* φ_p *and* $\varphi_{p'}$ *are conjugate in* $\mathrm{Bir}(\mathbb{A}^2)$.

Exercise 1.2. *Let* $\omega \in \mathbf{k}$ *be a primitive third root of unity.*

The map $g : (w : x : y : z) \mapsto (w : -x : \omega y : \omega^2 z)$ *acts on the cubic surface* $X_\lambda \subset \mathbb{P}^3$ *given by* $wx^2 + w^3 + y^3 + z^3 + \lambda wyz = 0$, *for any* $\lambda \in \mathbf{k}$.

The map $h : (w : x : y : z) \mapsto (w : -x : \omega y : \omega z)$ *acts on the cubic surface* $Y \subset \mathbb{P}^3$ *given by* $wx^2 + w^3 + y^3 + z^3 = 0$.

The exercise consists of deciding if the elements g *and* h *are conjugate, for some value of* λ. *By* conjugate, *we mean that there exists a birational map* $\varphi \colon X_\lambda \dashrightarrow Y$ *such that* $\varphi g = h\varphi$.

Hint: 1) *Look at the fixed points of the powers of* g *and* h*; you should be able to remove almost all values of* λ. 2) *Look at the actions of* g *and* h *on the curves that you obtained in* 1).

Exercise 1.3. *The map* $g : (w : x : y : z) \mapsto (w : \zeta^4 x : \zeta^3 y : \zeta^2 z)$ *acts on the cubic surface* $S \subset \mathbb{P}^3$ *given by* $w^3 + wx^2 + y^2 z + xz^2 = 0$, *where* ζ *is a primitive* 8*-th root of unity.*

Find a conic bundle structure $S \to \mathbb{P}^1$ *which is invariant by* g.

Exercise 1.4. *Is the birational map* $\alpha \in \mathrm{Bir}(\mathbb{A}^2)$ *given by*

$$\alpha \quad : (x,y) \dashrightarrow \left(\mathbf{i}x\,,\, \frac{(x+1)((\sqrt{2}-1)-x)y+(x^4-1)}{y+(x+1)((\sqrt{2}-1)-x))}\right).$$

conjugate to an element of $\mathrm{GL}(2,\mathbf{k})$ *?*

Hint: *compute the powers of* α.

2. Exercises that can be found in the notes

Exercise 1. *Let* X *be a smooth projective surface. Show that any point* $y \in \mathcal{B}(X)$ *is either a proper point or is in the* n*-th neighbourhood of a unique point* $x \in X$, *for some* $n \geq 1$. *If* $n > 1$, y *is in the first neighbourhood of a unique point* $y' \in \mathcal{B}(X)$ *which is in the* $(n-1)$*-th neighbourhood of* x.

Exercise 2. *For* $n \geq 1$, *prove that* $\mathrm{Aut}(\mathbb{F}_n)$ *is the quotient of* $\mathbf{k}^{n+1} \rtimes \mathrm{GL}(2,\mathbf{k})$ *by the subgroup of* $\mathrm{GL}(2,\mathbf{k})$ *consisting of diagonal matrices of the form* $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$, *where* $\mu^n = 1$. *(One way to do is to use induction on* n*: the case of* $\mathbb{F}_1$ *comes from* $\mathbb{P}^2$ *and the automorphisms of* $\mathbb{F}_{n+1}$ *that fix the base-point of* $(\varphi_n)^{-1}$ *correspond to automorphisms of* $\mathbb{F}_n$ *fixing the base-point of* φ_n*).*

Exercise 3. *Using the decomposition of birational maps from* $\mathbb{P}^2$ *to* $\mathbb{P}^2$ *into elementary links, shows that* $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by* $\mathrm{Aut}(\mathbb{P}^2)$ *and by the birational maps preserving a given pencil of lines through one point. Deduce the Noether-Castelnuovo theorem.*

Exercise 4. *On the del Pezzo surface* S *of degree* 6, *show that there is only one finite cyclic group* $G \subset \mathrm{Aut}(S)$ *such that* (G,S) *is minimal, up to conjugation in* $\mathrm{Aut}(S)$. *Show that this group is conjugate to a subgroup of* $\mathrm{Aut}(\mathbb{P}^2)$ *by a birational map* $S \dashrightarrow \mathbb{P}^2$. *(The answer to this question can be found in* [Bla09b, Lemma 9.7]*).*

Exercise 5. *Show that the cyclic subgroups* G *of order* 5 *of* $\mathrm{Aut}(S_5)$ *(which are all conjugate) are such that* (G,S) *is minimal. Prove that however* G *is conjugate to a subgroup of* $\mathrm{Aut}(\mathbb{P}^2)$ *by a birational map* $S \dashrightarrow \mathbb{P}^2$ *(as before, the answer can be found in* [Bla09b, Lemma 9.8]*)*

Exercise 6. *Using that a smooth cubic surface* S *is the blow-up of* 6 *points of* $\mathbb{P}^2$ *in general position, show that it contains exactly* 27 *lines, which are the* 27 (-1)*-curves of* S.

Exercise 7. *Let* $\alpha \in \mathbf{k}^*$ *be a* k*-th root of unity for some integer* $k \geq 2$. *Let* $g, h \in \mathrm{Aut}(\mathbb{P}^2)$ *be given by* $g \colon (x : y : z) \mapsto (x : \alpha y : \alpha^2 z)$ *and* $h \colon (x : y : z) \mapsto (x : \alpha y : z)$. *Prove that* g, h *are conjugate in* $\mathrm{Bir}(\mathbb{P}^2)$. *Compare the fixed locus of* g *and* h *when* $k \geq 3$.

References

[Bla09b] J. Blanc, *Linearisation of finite abelian subgroups of the Cremona group of the plane.* Groups Geom. Dyn. 3 (2009), no. 2, 215–266.